Global well-posedness of the spatially homogeneous Hubbard-Boltzmann equation

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Abstract

The Hubbard model is a simplified description for the evolution of interacting spin-$\frac{1}{2}$ fermions on a d-dimensional lattice. In a kinetic scaling limit, the Hubbard model can be associated with a matrix-valued Boltzmann equation, the Hubbard-Boltzmann equation. Its collision operator is a sum of two qualitatively different terms: The first term is similar to the collision operator of the fermionic Boltzmann-Nordheim equation. The second term leads to a momentum-dependent rotation of the spin basis. The rotation is determined by a principal value integral which depends quadratically on the state of the system and might become singular for non-smooth states. In this paper, we prove that the spatially homogeneous equation nevertheless has global solutions in $L^\infty(\mathbb{T}^d, \mathbb{C}^{2\times 2})$ for any initial data W_0 which satisfies the "Fermi constraint" in the sense that $0 \le W_0 \le 1$ almost everywhere. We also prove that there is a unique "physical" solution for which the Fermi constraint holds at all times. For the proof, we need to make a number of assumptions about the lattice dispersion relation which, however, are satisfied by the nearest neighbor Hubbard model, provided that $d \ge 3$. These assumptions suffice to guarantee that, although possibly singular, the local rotation term is generated by a function in $L^2(\mathbb{T}^d, \mathbb{C}^{2\times 2})$.

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1 Introduction

As discovered independently by Nordheim [1] and Peierls [2], the dynamics of weakly interacting quantum fluids can be well approximated by a kinetic equation of Boltzmann type. Mathematical properties of such quantum kinetic equations have been studied extensively, for a general review one can consult [3] and for recent results we refer to [4, 5, 6, 7, 8, 9, 10]. Of particular interest in our context is the work of Dolbeault [4] on the Boltzmann-Nordheim equation for spinless fermions.

In our contribution, we will study the kinetic equation derived from the fermionic Hubbard model, see [11] for details. Compared to the kinetic equation of [4], there are three important modifications. The Hubbard model describes the motion of electrons in a periodic background potential in the tight binding approximation, which means that the electrons move on the d-dimensional lattice $\mathbb{Z}^d$. In the spatially homogeneous case, the one considered here, this implies that the Wigner function $W(t,k)$ at time t is a function on the d-torus, $k \in \mathbb{T}^d$, the fundamental zone for the discrete Fourier transform. Secondly, since electrons have spin $\frac{1}{2}$, the Wigner function $W(t,k)$ depends on the spin and so naturally forms a 2×2-matrix. Thus the kinetic equation governs the evolution of a matrix-valued function on the torus for which, in general, $[W(t,k_1), W(t,k_2)] \neq 0$ if $k_1 \neq k_2$. Hence the conventional arguments have to be reworked.

The third modification is an appearance of a Vlasov type term in the Boltzmann equation. In the Hubbard model, the electron interaction is on-site and independent of spin. Hence, the microscopic Hamiltonian is invariant under global spin rotations. As a consequence, besides a conventional collision operator, the Hubbard-Boltzmann equation contains a term similar to that of the Vlasov equation. The new term rotates the k-dependent spin basis but does not generate entropy. At the first sight, the Vlasov term appears innocuous but, in fact, it is one major obstacle to overcome before arriving at a well-posed quantum kinetic equation: the term is defined by a principal value integral which might generate singularities even for regular initial data.

In the spatially homogeneous case, the Hubbard-Boltzmann equation reads

$$\partial_t W(t,k) = \mathcal{C}[W(t,\cdot)](k)\,, \qquad \mathcal{C}[W] := \mathcal{C}_{\text{diss}}[W] + \mathcal{C}_{\text{cons}}[W]\,. \tag{1.1}$$

Here $W = W_{ij}(t,k)$ is a 2×2 Hermitian matrix at time $t \geq 0$ with wave number $k \in \mathbb{T}^d$ and $i,j \in \{1,2\}$ counting for the spin-degrees of freedom. Physically, W describes the average density of electrons at wave number k with a spin density matrix $W(t,k)/\operatorname{tr}(W(t,k))$. We wish to solve this equation subject to an initial condition

$$W(0,k) = W_0(k)\,, \qquad \text{for all } k \in \mathbb{T}^d\,, \tag{1.2}$$

i.e., as a Cauchy problem with initial data W_0. The collision operator $\mathcal{C}[W]$ is a sum of a dissipative term $\mathcal{C}_{\text{diss}}[W]$ and a conservative Vlasov type term $\mathcal{C}_{\text{cons}}[W]$. Their properties depend crucially on the dispersion relation $\omega : \mathbb{T}^d \to \mathbb{R}$ of the waves generated by the quadratic term of the tight-binding Hamiltonian. For instance, in the Hubbard model on a d-dimensional square lattice, $\omega(k) = -\sum_{\nu=1}^{d} \cos(2\pi k^\nu)$.

The *dissipative part* of the collision operator is given by

$$\mathcal{C}_{\mathrm{diss}}[W](k_1) := \pi \int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \, \delta(\underline{k})\delta(\underline{\omega}) \\ \times \Big(\tilde{W}_1 W_3 J[\tilde{W}_2 W_4] + J[W_4 \tilde{W}_2] W_3 \tilde{W}_1 - W_1 \tilde{W}_3 J[W_2 \tilde{W}_4] - J[\tilde{W}_4 W_2] \tilde{W}_3 W_1 \Big) \,, \tag{1.3}$$

where we employ the following notations

$$\underline{k} := k_1 + k_2 - k_3 - k_4 \,, \quad \underline{\omega} := \omega_1 + \omega_2 - \omega_3 - \omega_4 \,, \quad \tilde{W} := 1 - W \,, \tag{1.4}$$

$$\omega_i := \omega(k_i) \,, \quad W_i := W(k_i) \,, \quad \tilde{W}_i := \tilde{W}(k_i) \,, \quad \text{for } i = 1,2,3,4 \,, \tag{1.5}$$

$$J[M] := 1 \operatorname{tr}(M) - M, \quad \text{for } M \in \mathbb{C}^{2\times 2} \,. \tag{1.6}$$

The shorthand notations $\underline{k}$ and $\underline{\omega}$ are somewhat rigid, as the dependence on variables k_i, $i = 1,2,3,4$, is implicit. We will only use them to shorten the notation for the collision kernels, in which case k_1 denotes always the fixed "input" variable and k_i, $i = 2,3,4$, the integration variables. The *conservative part* of the collision operator is written as a commutator with a W-dependent "effective Hamiltonian",

$$\mathcal{C}_{\mathrm{cons}}[W](k_1) := -\mathrm{i}[H_{\mathrm{eff}}[W](k_1), W(k_1)] \,, \tag{1.7}$$

where H_{eff} is defined formally as a principal value integral around $\underline{\omega} = 0$,

$$H_{\mathrm{eff}}[W](k_1) := \frac{1}{2} \, \mathrm{p.v.} \int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \, \delta(\underline{k}) \, \frac{1}{\underline{\omega}} \\ \times \Big(W_3 J[\tilde{W}_2 W_4] + J[W_4 \tilde{W}_2] W_3 + \tilde{W}_3 J[W_2 \tilde{W}_4] + J[\tilde{W}_4 W_2] \tilde{W}_3 \Big) \,. \tag{1.8}$$

The goal of our contribution is to establish that the evolution equation (1.1), together with (1.2), (1.3), (1.7) and (1.8), is well-posed for "fermionic initial data". Since the original Hubbard model describes fermions, the above Wigner matrix function at any time t needs to satisfy $0 \leq W(t,k) \leq 1$, as a matrix inequality[1] for almost every k; we call this property the *Fermi constraint*. Thus from the physics side, it is natural to look for solutions in the space of Lebesgue measurable functions $W(k)$ which satisfy the Fermi constraint almost everywhere. This requires to show that, if the initial data W_0 satisfies $0 \leq W_0(k) \leq 1$, then this property is propagated in time. For this purpose we use the approach of Dolbeault, somewhat modified to account for the matrix-valuedness of M and of the above constraints. Our construction relies heavily on the property that

$$AJ[BC] + CJ[BA] \geq 0 \,, \tag{1.9}$$

for arbitrary $n \times n$ matrices $A, B, C \geq 0$. (We thank David Reeb for most helpful discussions relating to the inequality (1.9).)

[1]We follow the convention that $M \geq 0$ if and only if M is a Hermitian matrix and its eigenvalues belong to $[0,\infty)$. We also recall that $M \in \mathbb{C}^{n\times n}$ satisfies $M \geq 0$ if and only if $(z, Mz) \geq 0$ for all $z \in \mathbb{C}^n$, and that this result is not valid if it is only checked for $z \in \mathbb{R}^n$.

For generic $W \in L^\infty(\mathbb{T}^d, \mathbb{C}^{2\times 2})$ we cannot expect that the principal value integral in (1.8) is convergent for all $k_1 \in \mathbb{T}^d$, or even that it converges almost everywhere to a bounded function. (Already the standard Hilbert-transform—a unitary operator on $L^2(\mathbb{R})$, defined via a similar principal value integral—offers such examples: the Hilbert transform of the characteristic function of any bounded interval has logarithmic divergences at both ends of the interval.) Of course, one could try to restrict the study to more regular function spaces in which the limit exists pointwise everywhere. However, there is no a priori reason why such a space would be invariant under the time-evolution. Our actual construction is rather indirect, but does achieve the desired goal. More precisely, our strategy is to prove first the well-posedness of a regularized problem for functions continuous in k, to solve $\partial_t W = \mathcal{C}_{\mathrm{diss}}[W] - \mathrm{i}[H^\varepsilon_{\mathrm{eff}}[W], W]$, and then to show that the solutions converge in L^2-norm to a unique solution of the original problem as the regulator is removed, $\varepsilon \to 0^+$.

In the well-studied continuum setup of Boltzmann equations, the dispersion relation is given by k^2, $k \in \mathbb{R}^d$, and the energy constraint $\underline{\omega} = 0$ has simple explicit solutions and one can integrate over both δ-functions to obtain an explicit integral operator involving only Lebesgue measures. In contrast, all lattice systems share the difficulty that even for the simplest lattice dispersion relations the solutions to $\underline{\omega} = 0$ are not easy to handle. Integration over $\delta(\underline{\omega})$ is even more problematic since the result typically involves singular integral kernels or might become ill-defined. A complete classification of the singularities resulting from such an integration over the δ-functions appears to be a hard problem in harmonic analysis and is certainly beyond the scope of the present paper.

Instead of a classification result, we provide here a set conditions for the dispersion relation ω under which both the dissipative collision integral and the principal value integral defining the effective Hamiltonian are sufficiently regular for the resulting solutions to be L^2-continuous. These assumptions and the main results are described in Sec. 2. In fact, for continuous W the dissipative part $\mathcal{C}_{\mathrm{diss}}$ can be defined even without the last (and the most complicated) of these conditions. We prove in Sec. 3 that then the formal integrations over the δ-functions in (1.3) can be replaced by an integration over a family of naturally defined bounded Borel measures. These measures are used in Sec. 4 to prove that the regularized problem is well-posed for continuous initial data. We have presented these results in somewhat greater generality than what is needed for the proof of the main theorem: such measures appear also in other phonon Boltzmann equations, and the properties proven in Sec. 3 could thus be of independent interest.

In Sec. 5, we show how the full set of assumptions leads to L^2-continuity of the collision operator and apply the resulting estimates to conclude the proof of the main theorem in Sec. 6. One technical problem in extending the phonon Boltzmann collision operator from continuous W to functions defined only Lebesgue almost everywhere is related to the measures derived in Sec. 3 which are singular with respect to the Lebesgue measure. Therefore, one needs to handle sets of measure zero carefully. In fact, it turns out that instead of working directly with integrals over the above Borel measures, it is better to define the collision integral via a limit procedure using L^2-approximants which are continuous in k (Lemma 3.3 and Corollary 3.4). These two definitions need not be pointwise equivalent, as the following example illustrates: Consider a function $f(x)$ which takes value one at $x = 0$

and is zero elsewhere. Then $\int \mathrm{d}x\, \delta(x-x_0) f(x) = 1$ at $x_0 = 0$ but f is L^2-equal to the zero map f_0 for which $\int \mathrm{d}x\, \delta(x-x_0) f_0(x) = 0$ for all x_0. However, should the two definitions of the collision operator disagree with each other, then the one presented here looks physically more reasonable: for instance, we prove here that it results in global well-posedness and in conservation of both total energy and spin. In fact, we show in Proposition 5.4 that the above definition of the collision integral can be simplified for dispersion relations satisfying all of the present assumptions. Then it can be defined for $W \in L^\infty$ as an L^2-limit of regularized Lebesgue integrals similar to those used in the definition of the principal value integral in H_{eff}.

To check that a given dispersion relation satisfies the required assumptions, is still a nontrivial problem in itself. We have included a proof in Appendix A which implies that our results indeed apply to the nearest neighbor square lattice hopping, at least as long as the dimension is high enough, $d \geq 3$.

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2 Main results

To give a proper mathematical definition of the collision operator $\mathcal{C}[W]$, we need to add some assumptions about the dispersion relation ω. For instance, if we would allow ω to be a constant map, we would have $\underline{\omega} = 0$ everywhere and thus $\delta(\underline{\omega}) = \infty$ identically. As in [12], we formulate our assumptions through properties of oscillatory integrals involving ω. There is considerable freedom, and the following choices are mainly made for convenience: (DR2) will imply that the term $\delta(\underline{\omega})$ leads to uniformly bounded measures and (DR3) will be used to prove the L^2-continuity of H_{eff}. Neither of these conditions is likely to be optimal, but they facilitate the technical estimates and are general enough to include many physically relevant cases. For instance, in Appendix A we show that nearest neighbor interactions on a square lattice with $d \geq 3$ satisfy all of the assumptions.

Whenever necessary, the d-torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ is understood as having the parameterization $[-\frac{1}{2}, \frac{1}{2}]^d$ with periodic identifications on the boundary. In particular, arithmetic is periodically extended to $\mathbb{T}^d$ and we have a normalization $\int_{\mathbb{T}^d} \mathrm{d}k = 1$.

Assumption 2.1 (Dispersion relation) *Suppose $d \geq 1$ and $\omega : \mathbb{T}^d \to \mathbb{R}$ satisfies all of the following:*

(DR1) *The periodic extension of* ω *is continuous and satisfies* $\omega(-k) = \omega(k)$.
(DR2) *(ℓ_3-dispersivity). Let us consider the* free propagator

$$p_t(x) = \int_{\mathbb{T}^d} \mathrm{d}k \, \mathrm{e}^{\mathrm{i}2\pi x \cdot k} \mathrm{e}^{-\mathrm{i}t\omega(k)} \,. \tag{2.1}$$

We assume that its ℓ_3*-norm belongs to* $L^3(\mathrm{d}t)$*, in other words, that* $t \mapsto \|p_t\|_3^3 \in L^1$.
(DR3) *(effective collisional dispersivity) For* $\sigma \in \{-1,1\}^4$ *and* $k, k' \in (\mathbb{T}^d)^3$ *define first*

$$\tilde{\Omega}(k;\sigma) := \sum_{i=1}^{3} \sigma_i \omega(k_i) + \sigma_4 \omega(k_1 + k_2 - k_3) \tag{2.2}$$

and then

$$\Omega_1(k,k';\sigma) := \tilde{\Omega}(k;\sigma)\,, \qquad \Omega_2(k,k';\sigma) := \tilde{\Omega}((k_1,k_2',k_3);\sigma)\,,$$
$$\Omega_3(k,k';\sigma) := \tilde{\Omega}(k';\sigma)\,, \qquad \Omega_4(k,k';\sigma) := \tilde{\Omega}((k_1',k_2,k_3');\sigma)\,.$$

Set for $s \in \mathbb{R}^4$, $\sigma \in \{-1,1\}^4$,

$$\mathcal{G}(s;\sigma) := \int_{(\mathbb{T}^d)^3 \times (\mathbb{T}^d)^3} \mathrm{d}^3k' \mathrm{d}^3k \, \mathrm{e}^{\mathrm{i}\sum_{i=1}^4 s_i \Omega_i(k,k';\sigma)} \,. \tag{2.3}$$

We assume that $C_{\mathcal{G}} := \max_\sigma \int_{\mathbb{R}^4} \mathrm{d}s |\mathcal{G}(s;\sigma)| < \infty$.

To study the solutions we consider the following function spaces

$$X_{\mathbb{H}} := \left\{ W \in C(\mathbb{T}^d, \mathbb{C}^{2\times 2}) \,\middle|\, W(k)^* = W(k) \text{ for all } k \in \mathbb{T}^d \right\}\,, \tag{2.4}$$
$$X_{\text{ferm}} := \left\{ W \in C(\mathbb{T}^d, \mathbb{C}^{2\times 2}) \,\middle|\, 0 \le W(k) \le 1 \text{ for all } k \in \mathbb{T}^d \right\}\,, \tag{2.5}$$
$$L^2_{\mathbb{H}} := \left\{ W \in L^2(\mathbb{T}^d, \mathbb{C}^{2\times 2}) \,\middle|\, W(k)^* = W(k) \text{ for a.e. } k \in \mathbb{T}^d \right\}\,, \tag{2.6}$$
$$L^2_{\text{ferm}} := \left\{ W \in L^2(\mathbb{T}^d, \mathbb{C}^{2\times 2}) \,\middle|\, 0 \le W(k) \le 1 \text{ for a.e. } k \in \mathbb{T}^d \right\}\,. \tag{2.7}$$

We equip the function space $X_{\mathbb{H}}$ with the sup-norm and the equivalence classes in $L^2_{\mathbb{H}}$ with L^2-norm which makes both spaces into *real* Banach spaces. It is easy to check that then X_{ferm} and L^2_{ferm} are closed convex subsets of $X_{\mathbb{H}}$ and $L^2_{\mathbb{H}}$, respectively.

To make the definitions of the norms explicit, we need to fix the matrix norm for $\mathbb{C}^{2\times 2}$. Since any two norms on a finite-dimensional vector space are equivalent, the choice does not play much role in the results. However, it will be convenient to consider here the Hilbert-Schmidt norm: we define for $M \in \mathbb{C}^{2\times 2}$

$$\|M\|^2 := \sum_{i,j=1}^{2} |M_{ij}|^2 \,. \tag{2.8}$$

For the matrix product this implies an estimate $\|M'M\| \le \|M'\|\|M\|$. The resulting L^2- and L^∞-norms in $L^2_{\mathbb{H}}$ are denoted by $\|W\|_2$ and $\|W\|_\infty$; explicitly, $\|W\|_2^2 =$

$\int_{\mathbb{T}^d} \mathrm{d}k\ \mathrm{tr}(W(k)^* W(k))$ and $\|W\|_\infty = \operatorname{ess\,sup}_{k\in\mathbb{T}^d} \sqrt{\mathrm{tr}(W(k)^* W(k))}$, where the essential supremum refers to the Lebesgue measure.

Our main results are summarized in the following three theorems. The first two give a precise meaning to the formal notations used in the definition of the collision operator. In the first theorem we explain how $\mathcal{C}_{\rm diss}$ is connected to a natural bounded Borel measure.

Theorem 2.2 *Suppose ω satisfies the conditions (DR1) and (DR2). In the collision operator, for a given $k_1 \in \mathbb{T}^d$, the notation $\mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\, \delta(\underline{k})\delta(\underline{\omega})$ refers to a regular complete bounded positive measure on $(\mathbb{T}^d)^3$ whose σ-algebra contains Borel sets and which satisfies for any $F \in C((\mathbb{T}^d)^3)$*

$$\int_{(\mathbb{T}^d)^3} \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\, \delta(\underline{k})\delta(\underline{\omega}) F(k_2,k_3,k_4) = \lim_{\varepsilon\to 0^+} \int_{(\mathbb{T}^d)^2} \frac{\mathrm{d}k_2\mathrm{d}k_3}{\pi} F(k_2,k_3,k_1+k_2-k_3) \frac{\varepsilon}{\varepsilon^2+\underline{\omega}^2}\,. \tag{2.9}$$

If $W \in X_{ferm}$, then $\mathcal{C}_{\rm diss}[W]$ is defined using this measure in (1.3).

For $W \in L^2_{ferm}$, $\mathcal{C}_{\rm diss}[W] \in L^2_{\mathbb{H}}$ is defined by using continuous approximants, as explained in detail in Corollary 3.4. If all of the conditions (DR1)–(DR3) hold, then for any $W \in L^2_{ferm}$

$$\begin{aligned}\mathcal{C}_{\rm diss}[W](k_1) = \lim_{\varepsilon\to 0^+} &\int_{(\mathbb{T}^d)^3} \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\, \delta(\underline{k}) \frac{\varepsilon}{\varepsilon^2+\underline{\omega}^2} \\ &\times \Big(\tilde{W}_1 W_3 J[\tilde{W}_2 W_4] + J[W_4\tilde{W}_2] W_3 \tilde{W}_1 - W_1\tilde{W}_3 J[W_2\tilde{W}_4] - J[\tilde{W}_4 W_2]\tilde{W}_3 W_1\Big)\,,\end{aligned} \tag{2.10}$$

where the limit is taken in $L^2_{\mathbb{H}}$-norm.

The second result shows that $H_{\rm eff}$ can be defined as an "L^2-principal value integral" which can also be obtained as a limit of terms using more regular cutoffs. (In fact, the second regularization arises naturally in the derivation of the equation from the microscopic Hubbard model, cf. [11].)

Theorem 2.3 *Suppose ω satisfies all of the conditions (DR1)–(DR3) and define for $\varepsilon > 0$*

$$\begin{aligned}H^\varepsilon_{\rm eff}[W](k_1) := \frac{1}{2} &\int_{(\mathbb{T}^d)^3} \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\, \delta(\underline{k}) \frac{\underline{\omega}}{\underline{\omega}^2+\varepsilon^2} \\ &\times \Big(W_3 J[\tilde{W}_2 W_4] + J[W_4\tilde{W}_2] W_3 + \tilde{W}_3 J[W_2\tilde{W}_4] + J[\tilde{W}_4 W_2]\tilde{W}_3\Big)\,.\end{aligned} \tag{2.11}$$

Then, given $W \in L^2_{ferm}$ there is an $L^2_{\mathbb{H}}$-limit of $H^\varepsilon_{\rm eff}[W]$ as $\varepsilon \to 0^+$ which we denote by $H_{\rm eff}[W]$. In addition, for any $W \in L^2_{ferm}$ we can find a sequence $\varepsilon_n \to 0^+$ such that for almost every k_1

$$\begin{aligned}H_{\rm eff}[W](k_1) = \lim_{n\to\infty} \frac{1}{2} &\int_{(\mathbb{T}^d)^3} \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\, \delta(\underline{k}) \frac{\mathbb{1}(|\underline{\omega}| \ge \varepsilon_n)}{\underline{\omega}} \\ &\times \Big(W_3 J[\tilde{W}_2 W_4] + J[W_4\tilde{W}_2] W_3 + \tilde{W}_3 J[W_2\tilde{W}_4] + J[\tilde{W}_4 W_2]\tilde{W}_3\Big)\,,\end{aligned} \tag{2.12}$$

where $W_i := W(k_i)$, $i = 1,2,3,4$.

The third theorem shows that, endowed with the above definitions, the Hubbard-Boltzmann equation is well-posed for any fermionic initial data. It also implies that the solution satisfies the basic properties expected from a kinetic scaling limit of the original Hubbard model: preservation of the Fermi property and of total energy and spin.

Theorem 2.4 *Suppose ω satisfies all of the conditions (DR1)–(DR3) and define $\mathcal{C}_{\rm diss}$ and $H_{\rm eff}$ as in Theorems 2.2 and 2.3, respectively. If $W_0 \in L^2_{ferm}$, then there is a unique $W \in C^{(1)}([0,\infty), L^2_{ferm})$ such that $W(0,k) = W_0(k)$ almost everywhere and for all $t > 0$*

$$\partial_t W_t = \mathcal{C}_{\rm diss}[W_t] - {\rm i}[H_{\rm eff}[W_t], W_t]\,, \tag{2.13}$$

where $W_t(k) := W(t,k)$. Here W_t depends $L^2_{\mathbb{H}}$-continuously on W_0 and the dependence is uniform on any compact interval of $[0,\infty)$. In addition, for all $t \geq 0$

$$\int_{\mathbb{T}^d} {\rm d}k\, \omega(k)\, {\rm tr}\, W(t,k) = \int_{\mathbb{T}^d} {\rm d}k\, \omega(k)\, {\rm tr}\, W_0(k)\,, \tag{2.14}$$

$$\int_{\mathbb{T}^d} {\rm d}k\, W(t,k) = \int_{\mathbb{T}^d} {\rm d}k\, W_0(k) \in \mathbb{C}^{2\times 2}\,. \tag{2.15}$$

Here the notation $C^{(1)}([0,\infty), L^2_{\rm ferm})$ denotes continuous functions $[0,\infty) \to L^2_{\rm ferm} \subset L^2_{\mathbb{H}}$ which are continuously Fréchet differentiable on $(0,\infty)$, considered as an open subset of the Banach space $\mathbb{R}$, assuming also that the limit $t \to 0^+$ of the derivatives exists. We recall here the basic property that if $W \in C^{(1)}([0,\infty), L^2_{\rm ferm})$ then for all $0 \leq t_0 \leq t_1 < \infty$

$$W_{t_1} = W_{t_0} + \int_{t_0}^{t_1} {\rm d}s\, \partial_s W_s\,, \tag{2.16}$$

where the integral is understood as a vector valued integral in $L^2_{\mathbb{H}}$ over the compact set $[t_0, t_1]$ and $\partial_s W_s \in L^2_{\mathbb{H}}$ denotes the Fréchet derivative at $s \in [t_0, t_1]$. (Strictly speaking, the Fréchet derivative should here be an element $D \in \mathcal{B}(\mathbb{R}, L^2_{\mathbb{H}})$. However, the action of this map is uniquely determined by giving $D1 \in L^2_{\mathbb{H}}$ and we always identify D with $D1$ for such maps and denote this by $\partial_s W_s$. Also, such a map is continuously Fréchet differentiable if and only if the map $s \mapsto \partial_s W_s$ is continuous which shows that the above vector valued integral in $L^2_{\mathbb{H}}$ is well-defined for any $W \in C^{(1)}([0,\infty), L^2_{\rm ferm})$. After these preliminaries, (2.16) follows from the fundamental theorem of calculus and the easily checked property that $\partial_s \Lambda[W_s] = \Lambda[\partial_s W_s]$ for any dual element $\Lambda \in (L^2_{\mathbb{H}})^*$.)

3 Basic properties of the dissipative term $\mathcal{C}_{\rm diss}$

This section concerns the definition of the dissipative part of the collision operator. In particular, our goal is to show that the first two assumptions, (DR1) and (DR2), suffice to define the dissipative term $\mathcal{C}_{\rm diss}[W]$ via a Borel measure with an L^2-continuity property which will be needed later.

We begin by showing that the measure merits its symbolic notation which uses the two formal delta-functions, at least as long as the integrand is continuous. This is the main goal of the first Proposition here. The result is somewhat more general than what we need for the present proof but the apparently superfluous properties could well become useful in rigorous studies of other phonon Boltzmann equations. For the statement, we consider more general "energy constraints": define for $k \in (\mathbb{T}^d)^4$ and $\sigma \in \{-1,1\}^4$

$$\Omega(k,\sigma) := \sum_{i=1}^{4} \sigma_i \omega(k_i)\,. \tag{3.1}$$

This is related to the definition in (2.2) by $\Omega(k,\sigma) = \tilde{\Omega}((k_1,k_1,k_2),\sigma)$, whenever $k_1 + k_2 = k_3 + k_4 \pmod 1$. In addition, the combination appearing in the definition of the collision operator satisfies $\underline{\omega} = \Omega(k,(1,1,-1,-1))$.

Proposition 3.1 *Assume that $\omega : \mathbb{T}^d \to \mathbb{R}$ is continuous and satisfies (DR2), and define $\tilde{\Omega}$ and Ω as in (2.2) and (3.1). Then for every $k_1 \in \mathbb{T}^d$, $\alpha \in \mathbb{R}$, and $\sigma \in \{-1,1\}^4$ the map*

$$C((\mathbb{T}^d)^3) \ni F \mapsto \lim_{\varepsilon\to 0^+} \int_{(\mathbb{T}^d)^2} \frac{\mathrm{d}k_2 \mathrm{d}k_3}{\pi} F(k_2,k_3,k_1+k_2-k_3) \frac{\varepsilon}{\varepsilon^2 + (\tilde{\Omega}((k_1,k_2,k_3),\sigma)-\alpha)^2}\,, \tag{3.2}$$

defines a regular complete bounded positive measure on $(\mathbb{T}^d)^3$, σ-algebra containing Borel sets, which we denote by $\nu_{k_1,\alpha,\sigma}(\mathrm{d}^3 k)$ or $\int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4\, \delta(\Omega(k,\sigma)-\alpha)\delta(k_1+k_2-k_3-k_4)$.

The measure has the following properties, for any fixed σ,

1. *Define $\sigma_{\mathrm{coll}}(k_1,\alpha;\sigma) := \int \mathrm{d}\nu_{k_1,\alpha,\sigma}$. Then $\sigma_{\mathrm{coll}} \in C(\mathbb{T}^d \times \mathbb{R})$.*
2. *The set $\{(k_2,k_3,k_4) \in (\mathbb{T}^d)^3 \,|\, k_1+k_2-k_3-k_4 = 0,\ \Omega(k,\sigma) = \alpha\}$ contains the support of the measure $\nu_{k_1,\alpha,\sigma}$.*
3. *Suppose $F \in C((\mathbb{T}^d)^3)$, $f \in C(\mathbb{R})$, $k_1 \in \mathbb{T}^d$, and $\alpha \in \mathbb{R}$ are given. Then*
$$\int \nu_{k_1,\alpha,\sigma}(\mathrm{d}^3 k')\, f(\Omega((k_1,k'),\sigma))F(k') = f(\alpha) \int \nu_{k_1,\alpha,\sigma}(\mathrm{d}^3 k')\, F(k')\,. \tag{3.3}$$
In particular, $\int \nu_{k_1,0,\sigma}(\mathrm{d}^3 k')\, \Omega((k_1,k'),\sigma)F(k') = 0$ for any $F \in C((\mathbb{T}^d)^3)$ and all $k_1 \in \mathbb{T}^d$.
4. *If $G \in C((\mathbb{T}^d)^4 \times \mathbb{R})$, then the map $(k_1,\alpha) \mapsto \int \nu_{k_1,\alpha,\sigma}(\mathrm{d}^3 k')\, G(k_1,k',\alpha)$ is continuous on $\mathbb{T}^d \times \mathbb{R}$.*
5. *Suppose $F \in C((\mathbb{T}^d)^3 \times \mathbb{R})$ and $m > \sup_k |\tilde{\Omega}(k,\sigma)|$. Then for all $k_1 \in \mathbb{T}^d$ we have $\int_{-m}^{m} \mathrm{d}\alpha \left(\int \nu_{k_1,\alpha,\sigma}(\mathrm{d}^3 k')\, F(k',\alpha)\right) = \int_{(\mathbb{T}^d)^2} \mathrm{d}k_2 \mathrm{d}k_3\, F(k_2,k_3,k_1+k_2-k_3,\tilde{\Omega}((k_1,k_2,k_3),\sigma))$.*

Proof: σ will be considered fixed in the following, and we drop the dependence on it from the notation here. Define for $x \in \mathbb{R}$ and $\varepsilon > 0$

$$\varphi^0_\varepsilon(x) := \frac{\varepsilon}{x^2+\varepsilon^2} \quad \text{and} \quad \hat{\varphi}^0_\varepsilon(x) := \frac{1}{2}\mathrm{e}^{-\varepsilon|x|}\,. \tag{3.4}$$

Then $\hat{\varphi}^0_\varepsilon \in L^1 \cap L^\infty$, and it is straightforward to check that pointwise for *all* $x \in \mathbb{R}$ we have $\varphi^0_\varepsilon(x) = \int_{-\infty}^{\infty} \mathrm{d}s\, \hat{\varphi}^0_\varepsilon(s)\mathrm{e}^{\mathrm{i}sx}$. We need to consider the maps

$$\Lambda_{k_1,\alpha,\varepsilon}[F] := \int_{(\mathbb{T}^d)^2} \frac{\mathrm{d}k_2\mathrm{d}k_3}{\pi}\, F(k_2,k_3,k_1+k_2-k_3)\varphi^0_\varepsilon(\tilde{\Omega}((k_1,k_2,k_3),\sigma)-\alpha) \tag{3.5}$$

which are all obviously continuous linear functionals on $C((\mathbb{T}^d)^3)$ with

$$\|\Lambda_{k_1,\alpha,\varepsilon}\| \le \int_{(\mathbb{T}^d)^2} \frac{\mathrm{d}k_2\mathrm{d}k_3}{\pi}\, \varphi^0_\varepsilon(\tilde{\Omega}((k_1,k_2,k_3),\sigma)-\alpha) =: C_{k_1,\alpha,\varepsilon}\,. \tag{3.6}$$

In addition, Fubini's theorem yields for any $F \in C((\mathbb{T}^d)^3)$

$$\Lambda_{k_1,\alpha,\varepsilon}[F] = \int_{-\infty}^{\infty} \mathrm{d}s\, \hat{\varphi}^0_\varepsilon(s) \int_{(\mathbb{T}^d)^2} \frac{\mathrm{d}k_2\mathrm{d}k_3}{\pi}\, F(k_2,k_3,k_1+k_2-k_3)\mathrm{e}^{\mathrm{i}s\tilde{\Omega}((k_1,k_2,k_3),\sigma)-\mathrm{i}s\alpha}\,. \tag{3.7}$$

Assume first that F is a trigonometric polynomial, i.e., that there is $f : (\mathbb{Z}^d)^3 \to \mathbb{C}$ which has finite support and

$$F(k) = \sum_{x\in(\mathbb{Z}^d)^3} \mathrm{e}^{-\mathrm{i}2\pi k\cdot x} f(x)\,. \tag{3.8}$$

Then

$$\begin{aligned}
&\int_{(\mathbb{T}^d)^2} \mathrm{d}k_2\mathrm{d}k_3\, F(k_2,k_3,k_1+k_2-k_3)\mathrm{e}^{\mathrm{i}s\tilde{\Omega}((k_1,k_2,k_3),\sigma)-\mathrm{i}s\alpha} \\
&\quad = \sum_{x\in(\mathbb{Z}^d)^3} \mathrm{e}^{\mathrm{i}s\sigma_1\omega(k_1)-\mathrm{i}s\alpha} f(x) \int_{(\mathbb{T}^d)^2} \mathrm{d}k_2\mathrm{d}k_3\, \mathrm{e}^{\mathrm{i}s(\sigma_2\omega(k_2)+\sigma_3\omega(k_3)+\sigma_4\omega(k_1+k_2-k_3))} \\
&\qquad \times \mathrm{e}^{-\mathrm{i}2\pi(k_2\cdot(x_1+x_3)+k_3\cdot(x_2-x_3)+k_1\cdot x_3)}\,.
\end{aligned} \tag{3.9}$$

The remaining convolution integral can be expressed in terms of $p_t(x)$, which is the inverse Fourier transform of $k \mapsto \mathrm{e}^{-\mathrm{i}t\omega(k)}$. Using Parseval's theorem to the k_3-integral and then Fubini's theorem proves that (3.9) is equal to

$$\sum_{x\in(\mathbb{Z}^d)^3} \mathrm{e}^{\mathrm{i}s\sigma_1\omega(k_1)-\mathrm{i}s\alpha} f(x) \sum_{y\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}2\pi k_1\cdot(y+x_2)} p_{-\sigma_2 s}(-y-x_1-x_2)p_{-\sigma_4 s}(y+x_2-x_3)p_{-\sigma_3 s}(y)\,. \tag{3.10}$$

Thus by Hölder's inequality, the property $\|p_{-s}\|_3 = \|p_s\|_3$, and using (DR2), its absolute value is bounded by $\|f\|_1 \|p_s\|_3^3 \in L^1(\mathrm{d}s)$. Therefore, dominated convergence can be used here to prove that, when $\varepsilon \to 0^+$,

$$\Lambda_{k_1,\alpha,\varepsilon}[F] \to \int_{-\infty}^{\infty} \mathrm{d}s \left(\int_{(\mathbb{T}^d)^2} \frac{\mathrm{d}k_2 \mathrm{d}k_3}{2\pi} \, F(k_2, k_3, k_1 + k_2 - k_3) \mathrm{e}^{\mathrm{i}s\tilde{\Omega}((k_1,k_2,k_3),\sigma) - \mathrm{i}s\alpha} \right) . \tag{3.11}$$

In addition, dominated convergence also implies that the limit defines a continuous function of (k_1, α), as both F and $\tilde{\Omega}$ are continuous. Applying the bounds for $f(x) = \mathbb{1}(x = 0)$, i.e., for $F(k) = 1$, also proves that there is $C' < \infty$, independent of α, k_1 and ε, such that $C_{k_1,\alpha,\varepsilon} \le C'$.

Therefore, we have now proven that $(\Lambda_{k_1,\alpha,\varepsilon})_{\varepsilon>0}$, with α, k_1 fixed, form an equicontinuous family of linear functionals on $C((\mathbb{T}^d)^3)$ which converges at any F which is a trigonometric polynomial. Since trigonometric polynomials are dense in $C((\mathbb{T}^d)^3)$ by the Stone-Weierstrass theorem, this implies that the family converges then for any $F \in C((\mathbb{T}^d)^3)$ and the limit defines a unique $\Lambda_{k_1,\alpha} \in C((\mathbb{T}^d)^3)^*$. (Such a statement is true for any equicontinuous sequence of linear mappings between Banach spaces; for a more generic statement, see for instance [13, Exercise 2.14]. By (3.11), the limits obtained from an arbitrary sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ agree on a dense set, and therefore they must all be equal. We denote the common limit by $\Lambda_{k_1,\alpha}$ and it follows that $\lim_{\varepsilon\to 0^+} \Lambda_{k_1,\alpha,\varepsilon}[F] = \Lambda_{k_1,\alpha}[F]$ for all continuous F, even though it could well happen that the integral on the right hand side of (3.11) is not absolutely convergent for some F.) Since obviously $\Lambda_{k_1,\alpha}[F] \ge 0$ for any $F \ge 0$ and $(\mathbb{T}^d)^3$ is compact, Riesz representation theorem implies that there is a complete regular positive measure $\nu_{k_1,\alpha}$ such that its σ-algebra contains Borel sets and $\Lambda_{k_1,\alpha}[F] = \int \mathrm{d}\nu_{k_1,\alpha} F$ for $F \in C((\mathbb{T}^d)^3)$. In addition, since $\sigma_{\mathrm{coll}}(k_1, \alpha) = \int \mathrm{d}\nu_{k_1,\alpha} = \lim_{n\to\infty} C_{k_1,\alpha,1/n} \le C'$, the measure is bounded and σ_{coll} is a continuous function of k_1, α. We have thus proven the first part of the Proposition, and item 1 in the second part.

To prove item 2, fix k_1, denote $\nu := \nu_{k_1,\alpha}$ and assume that

$$\tilde{k} \notin S := \left\{ k \in (\mathbb{T}^d)^{\{2,3,4\}} \,\middle|\, k_1 + k_2 - k_3 - k_4 = 0,\ \Omega(k, \sigma) = \alpha \right\} . \tag{3.12}$$

Since ω is continuous, S is closed, and thus there is $\delta > 0$ such that $B(\tilde{k}, 2\delta) \subset S^c$. We can then choose a sequence of continuous approximants $\phi_n \in C((\mathbb{T}^d)^3)$ of the characteristic function of the open ball $B(\tilde{k}, \delta/2)$ such that the sequence converges pointwise, has values in $[0, 1]$ and is equal to 0 for any k' with $|k' - \tilde{k}| \ge \delta$. By dominated convergence, then $\nu(B(\tilde{k}, \delta/2)) = \lim_{n\to\infty} \int \nu(\mathrm{d}^3 k') \phi_n(k')$. Here

$$\int \nu(\mathrm{d}^3 k') \, \phi_n(k') = \lim_{\varepsilon\to 0^+} \int_{(\mathbb{T}^d)^2} \frac{\mathrm{d}k_2 \mathrm{d}k_3}{\pi} \, \phi_n(k_2, k_3, k_1 + k_2 - k_3) \frac{\varepsilon}{\varepsilon^2 + (\tilde{\Omega}((k_1, k_2, k_3), \sigma) - \alpha)^2} . \tag{3.13}$$

Let S_1 denote the support of the function $(k_2, k_3) \mapsto \phi_n(k_2, k_3, k_1 + k_2 - k_3)$. If S_1 is empty, then the map is zero and $\int \nu(\mathrm{d}^3 k') \, \phi_n(k') = 0$. Assume thus $S_1 \ne \emptyset$. If $(k_2, k_3) \in S_1$, then $k' := (k_2, k_3, k_1 + k_2 - k_3) \in \operatorname{supp} \phi_n$, and thus $k' \notin S$ and therefore $\tilde{\Omega}((k_1, k_2, k_3), \sigma) =$

$\Omega((k_1,k'),\sigma)\neq\alpha$. Since S_1 is compact and $(k_2,k_3)\mapsto|\tilde\Omega((k_1,k_2,k_3),\sigma)-\alpha|$ is continuous, there is a minimum c_0 of the map on S_1, and $c_0>0$. But then the integrand in (3.13) is bounded by ε/c_0^2, which implies that $\int\nu(\mathrm{d}^3k')\,\phi_n(k')=0$. Therefore, $\nu(B(\tilde k,\delta/2))=0$ and thus $\tilde k$ is not in the support of $\nu_{k_1,\alpha}$. This proves item 2.

Item 3 is then a consequence of the fact that, since the support of $\nu_{k_1,\alpha}$ is part of the Borel set S, necessarily for any continuous function G one has $\int\mathrm{d}\nu_{k_1,\alpha}G=\int_S\mathrm{d}\nu_{k_1,\alpha}G$. To prove item 4, consider the map $I:(k_1,\alpha)\mapsto\int\nu_{k_1,\alpha,\sigma}(\mathrm{d}^3k')\,G(k_1,k',\alpha)$ for some $G\in C((\mathbb{T}^d)^4\times\mathbb{R})$. Set $r:=1+\sup_k|\tilde\Omega(k,\sigma)|$, which is finite since $\tilde\Omega$ is continuous, and choose a function $f\in C(\mathbb{R})$ such that $0\le f\le 1$ and $f(x)=1$ for all $|x|\le r-1$ and $f(x)=0$ for all $|x|\ge r$. Define $\bar G(k,\alpha):=f(\alpha)G(k,\alpha)$. By the Stone-Weierstrass theorem, we can find a sequence $\psi_n(k,\alpha)$ such that it converges uniformly to $\bar G$ on $(\mathbb{T}^d)^4\times[-r,r]$ and each ψ_n is a linear combination of functions of the form $\mathrm{e}^{-\mathrm{i}\pi x_0\cdot\alpha/r}\mathrm{e}^{-\mathrm{i}2\pi\sum_{i=1}^4 k_i\cdot x_i}$ with $x_j\in\mathbb{Z}^d$, $j=0,1,\ldots,4$. Clearly, $I(k_1,\alpha)=\int\nu_{k_1,\alpha,\sigma}(\mathrm{d}^3k')\,\bar G(k_1,k',\alpha)$ if $|\alpha|\le r-1$, but this in fact holds for all α since for $|\alpha|>r-1$ we have $\int\nu_{k_1,\alpha,\sigma}(\mathrm{d}^3k')\,\bar G(k_1,k',\alpha)=0=I(k_1,\alpha)$ by item 2. By the previous results, each of the functions $(k_1,\alpha)\mapsto\int\nu_{k_1,\alpha,\sigma}(\mathrm{d}^3k')\,\psi_n(k_1,k',\alpha)$ is continuous and hence so is I on $(\mathbb{T}^d)^4\times[-r,r]$. However, $I(k_1,\alpha)=0$ for all $|\alpha|\ge r$, and thus I is everywhere continuous.

To prove item 5, suppose $F\in C((\mathbb{T}^d)^3\times\mathbb{R})$ and $m>\sup_k|\tilde\Omega(k,\sigma)|$, and fix $k_1\in\mathbb{T}^d$. By the previous results, we can then apply dominated convergence and Fubini's theorem and conclude that

$$\begin{aligned}&\int_{-m}^{m}\mathrm{d}\alpha\left(\int\nu_{k_1,\alpha,\sigma}(\mathrm{d}^3k')\,F(k',\alpha)\right)\\&\quad=\lim_{\varepsilon\to0^+}\int_{(\mathbb{T}^d)^2}\frac{\mathrm{d}k_2\mathrm{d}k_3}{\pi}\int_{-m}^{m}\mathrm{d}\alpha\,F(k_2,k_3,k_1+k_2-k_3,\alpha)\frac{\varepsilon}{\varepsilon^2+(\tilde\Omega((k_1,k_2,k_3),\sigma)-\alpha)^2}\,.\end{aligned}\tag{3.14}$$

We change the integration variable α to $s=(\tilde\Omega-\alpha)/\varepsilon$, where $\tilde\Omega:=\tilde\Omega((k_1,k_2,k_3),\sigma)$. The resulting integral is $\int_{(\tilde\Omega-m)/\varepsilon}^{(\tilde\Omega+m)/\varepsilon}\mathrm{d}s\,\frac{1}{1+s^2}F(k_2,k_3,k_1+k_2-k_3,\tilde\Omega-\varepsilon s)$ which is uniformly bounded and, since $|\tilde\Omega|<m$, it approaches $\pi F(k_2,k_3,k_1+k_2-k_3,\tilde\Omega)$ when $\varepsilon\to0^+$. Dominated convergence can thus be applied to conclude that $\int_{-m}^{m}\mathrm{d}\alpha\left(\int\nu_{k_1,\alpha,\sigma}(\mathrm{d}^3k')\,F(k',\alpha)\right)=\int_{(\mathbb{T}^d)^2}\mathrm{d}k_2\mathrm{d}k_3\,F(k_2,k_3,k_1+k_2-k_3,\tilde\Omega((k_1,k_2,k_3),\sigma))$. This concludes the proof of the Proposition. □

In particular, the result thus implies that the (up to now formal) δ-functions in the definition of $\mathcal{C}_{\mathrm{diss}}$ correspond to a well-defined measure. From now on we denote it by ν_{k_1}, $k_1\in\mathbb{T}^d$, i.e., we set $\nu_{k_1}:=\nu_{k_1,0,(1,1,-1,-1)}$. The following results explain how we use it to define $\mathcal{C}_{\mathrm{diss}}[W]$ if $W\in L^2_{\mathrm{ferm}}$. This somewhat indirect construction appears necessary since $k_1\mapsto\int\nu_{k_1,0,\sigma}(\mathrm{d}^3k)F(k)$ might not even map sets of Lebesgue measure zero on $(\mathbb{T}^d)^3$ into sets of measure zero on $\mathbb{T}^d$. Nevertheless, Corollary 3.4 shows that the dissipative part of the collision operator, which only contains products of $L^\infty(\mathbb{T}^d)$-functions, can be meaningfully extended into a map from $(L^\infty(\mathbb{T}^d))^3$ to $L^\infty(\mathbb{T}^d)$.

We begin with a result which shows that "Fubini's theorem" works despite the δ-functions if the integrand is continuous.

Corollary 3.2 *Suppose that* $\omega : \mathbb{T}^d \to \mathbb{R}$ *satisfies (DR1) and (DR2) and suppose* $\alpha \in \mathbb{R}$ *and* $\sigma \in \{-1,1\}^4$ *are given. Let* $\nu_{k,\alpha,\sigma}$ *denote measures satisfying Proposition 3.1, and set* $\sigma^{(2)} := (\sigma_2, \sigma_1, \sigma_3, \sigma_4)$, $\sigma^{(3)} := (\sigma_3, \sigma_2, \sigma_1, \sigma_4)$ *and* $\sigma^{(4)} := (\sigma_4, \sigma_2, \sigma_1, \sigma_3)$*. If* $G : (\mathbb{T}^d)^4 \to \mathbb{C}$ *is continuous then*

$$\begin{aligned}
&\int_{\mathbb{T}^d} \mathrm{d}k_1 \left(\int \nu_{k_1,\alpha,\sigma}(\mathrm{d}^3k')\, G(k_1, k_1', k_2', k_3') \right) \\
&\quad = \int_{\mathbb{T}^d} \mathrm{d}k_2 \left(\int \nu_{k_2,\alpha,\sigma^{(2)}}(\mathrm{d}^3k')\, G(k_1', k_2, k_2', k_3') \right) . \\
&\quad = \int_{\mathbb{T}^d} \mathrm{d}k_3 \left(\int \nu_{k_3,\alpha,\sigma^{(3)}}(\mathrm{d}^3k')\, G(k_2', -k_1', k_3, -k_3') \right) . \\
&\quad = \int_{\mathbb{T}^d} \mathrm{d}k_4 \left(\int \nu_{k_4,\alpha,\sigma^{(4)}}(\mathrm{d}^3k')\, G(k_2', -k_1', -k_3', k_4) \right) .
\end{aligned} \tag{3.15}$$

Proof: Assume G to be continuous. By Proposition 3.1, all four of the above integrals are over continuous functions and hence well-defined, and we can apply dominated convergence and Fubini's theorem to conclude that

$$\begin{aligned}
&\int_{\mathbb{T}^d} \mathrm{d}k_4 \left(\int \nu_{k_4,\alpha,\sigma^{(4)}}(\mathrm{d}^3k')\, G(k_2', -k_1', -k_3', k_4) \right) \\
&\quad = \lim_{\varepsilon\to 0^+} \int_{(\mathbb{T}^d)^3} \frac{\mathrm{d}k_4 \mathrm{d}k_1' \mathrm{d}k_2'}{\pi} G(k_2', -k_1', k_2' - k_1' - k_4, k_4) \frac{\varepsilon}{\varepsilon^2 + (\tilde{\Omega}((k_4, k_1', k_2'), \sigma^{(4)}) - \alpha)^2} .
\end{aligned} \tag{3.16}$$

By Fubini's theorem, the value of the integral on the right hand side can be obtained also by iterating the three integrals in an arbitrary order. Choose to do k_4 first and change there the integration variable to $k_3 = k_2' - k_1' - k_4$. Then $k_4 = k_2' - k_1' - k_3$ and, since $\omega(-k) = \omega(k)$, also $\omega(k_4 + k_1' - k_2') = \omega(k_3)$. Do next k_1', and change the integration variable to $k_2 = -k_1'$. Rename the last integration variable to k_1. Then $\tilde{\Omega}((k_4, k_1', k_2'), \sigma^{(4)}) = \tilde{\Omega}((k_1, k_2, k_3), \sigma)$, and we can conclude that the integral is equal to

$$\int_{\mathbb{T}^d} \mathrm{d}k_1 \left(\int_{\mathbb{T}^d} \mathrm{d}k_2 \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}k_3}{\pi} G(k_1, k_2, k_3, k_1 + k_2 - k_3) \frac{\varepsilon}{\varepsilon^2 + (\tilde{\Omega}((k_1, k_2, k_3), \sigma) - \alpha)^2} \right) \right) . \tag{3.17}$$

Then, by applying Fubini's and dominated convergence theorem, as well as Proposition 3.1, we find that (3.16) is equal to $\int_{\mathbb{T}^d} \mathrm{d}k_1 \left(\int \nu_{k_1,\alpha,\sigma}(\mathrm{d}^3k')\, G(k_1, k_1', k_2', k_3') \right)$. This proves the equality of the first and last of the expressions in (3.15). The proofs that the other two expressions are equal to the first one are very similar, only simpler, and we skip them here. $\square$

Lemma 3.3 *Assume that ω satisfies (DR1) and (DR2). Consider arbitrary $w_i \in L^\infty(\mathbb{T}^d)$, $i = 1,2,3$, and sequences $w_{i,n} \in C(\mathbb{T}^d)$, $n \in \mathbb{N}$, such that $w_{i,n} \to w_i$ in L^2-norm and $|w_{i,n}(k)| \le \|w_i\|_\infty$ for all i,n,k. Then there is $\mathcal{C}_0 \in L^2(\mathbb{T}^d)$ such that the sequence of continuous functions $k_1 \mapsto \int_{(\mathbb{T}^d)^3} \nu_{k_1}(\mathrm{d}^3k') \prod_{i=1}^3 w_{i,n}(k_i')$ converges in L^2 to $\mathcal{C}_0$ as $n \to \infty$. In addition, there is a constant C, which depends only on ω, such that*

$$\|\mathcal{C}_0\|_{L^\infty} \le C \prod_{i=1}^3 \|w_i\|_{L^\infty}\,, \tag{3.18}$$

and, if $w_i' \in L^\infty(\mathbb{T}^d)$, $i = 1,2,3$, and $w_{i,n}' \in C(\mathbb{T}^d)$, $n \in \mathbb{N}$, are another collection of functions as above, and $\mathcal{C}_0'$ denotes the corresponding limit, then

$$\|\mathcal{C}_0 - \mathcal{C}_0'\|_{L^2} \le Cm^2 \sum_{i=1}^3 \|w_i - w_i'\|_{L^2}\,, \tag{3.19}$$

where $m := \max_i(\|w_i\|_\infty, \|w_i'\|_\infty)$.

Therefore, we can identify $\mathcal{C}_0$ with a unique map $(L^\infty(\mathbb{T}^d))^3 \to L^\infty(\mathbb{T}^d)$. This map satisfies all of the following properties:

1. *It is linear and commutes with complex conjugation, $\mathcal{C}_0[w_1^*, w_2^*, w_3^*] = \mathcal{C}_0[w_1,w_2,w_3]^*$.*
2. *The bounds (3.18) and (3.19) hold for $\mathcal{C}_0 = \mathcal{C}_0[w_1,w_2,w_3]$ and $\mathcal{C}_0' = \mathcal{C}_0[w_1',w_2',w_3']$.*
3. *If $w_i \in C(\mathbb{T}^d)$ for all i, then $\mathcal{C}_0[w_1,w_2,w_3] \in C(\mathbb{T}^d)$ and for every $k_1 \in \mathbb{T}^d$*

$$\mathcal{C}_0[w_1,w_2,w_3](k_1) = \int_{(\mathbb{T}^d)^3} \nu_{k_1}(\mathrm{d}^3k') \prod_{i=1}^3 w_i(k_i')\,. \tag{3.20}$$

Proof: Suppose $w_{i,n}$, $w_{i,n}'$ satisfy the assumptions of the Lemma. Since they are continuous, setting $g_{i,n}(k_1) := \int_{(\mathbb{T}^d)^3} \nu_{k_1}(\mathrm{d}^3k') \prod_{i=1}^3 w_{i,n}(k_i')$ and $g_{i,n}'(k_1) := \int_{(\mathbb{T}^d)^3} \nu_{k_1}(\mathrm{d}^3k') \prod_{i=1}^3 w_{i,n}'(k_i')$ yields continuous functions on $\mathbb{T}^d$, by Proposition 3.1. Denote $C := \sup_{k_1,\sigma'} \int \mathrm{d}\nu_{k_1,0,\sigma'}$, which is finite by Proposition 3.1, and set $m := \max_i(\|w_i\|_\infty, \|w_i'\|_\infty) < \infty$. Then we have the obvious bounds $|g_{i,n}(k_1)| \le C \prod_{i=1}^3 \|w_i\|_\infty$ and $|g_{i,n}'(k_1)| \le C \prod_{i=1}^3 \|w_i'\|_\infty$, and, by telescoping, we also find that for any k_1

$$|g_{i,n}(k_1) - g_{i,n}'(k_1)| \le m^2 \sum_{i=1}^3 \int_{(\mathbb{T}^d)^3} \nu_{k_1}(\mathrm{d}^3k') |w_{i,n}(k_i') - w_{i,n}'(k_i')|\,. \tag{3.21}$$

Hölder's inequality and $\int \mathrm{d}k_1 = 1$ imply that $\int \mathrm{d}k_1 \left(\int \nu_{k_1}(\mathrm{d}^3k')|w_{i,n}(k_i') - w_{i,n}'(k_i')|\right)^2 \le C \int \mathrm{d}k_1 \int \nu_{k_1}(\mathrm{d}^3k')|w_{i,n}(k_i') - w_{i,n}'(k_i')|^2$. By Corollary 3.2, the last expression is equal to $C \int \mathrm{d}k_1 |w_{i,n}(k_1) - w_{i,n}'(k_1)|^2 \int \nu_{k_1,0,\sigma^{(i+1)}}(\mathrm{d}^3k') \le C^2 \|w_{i,n} - w_{i,n}'\|_{L^2}^2$. Therefore,

$$\|g_{i,n} - g_{i,n}'\|_{L^2} \le Cm^2 \sum_{i=1}^3 \|w_{i,n} - w_{i,n}'\|_{L^2}\,. \tag{3.22}$$

Consider then some $n_0 \in \mathbb{N}$, and define $w''_{i,n} := w_{i,n+n_0}$. Then $w''_{i,n} \to w_i$ in L^2 and we can apply the above results to the sequences $w''_{i,n}$. Since then $g''_{i,n} = g_{i,n+n_0}$ and $w_{i,n} - w''_{i,n} \to 0$, the bound in (3.22) proves that $g_{i,n}$ is a Cauchy sequence in L^2. Hence the L^2-limit $\mathcal{C}_0$ exists and there is a subsequence (n_ℓ) such that $\mathcal{C}_0(k) := \lim_{\ell\to\infty} g_{i,n_\ell}(k)$ for Lebesgue almost every $k \in \mathbb{T}^d$. At every such point we thus have $|\mathcal{C}_0(k)| \le C \prod_{i=1}^3 \|w_i\|_\infty$. Hence (3.18) holds, and we have proven the first part of the Lemma.

These results can also be applied to the sequences $w'_{i,n}$, and the corresponding limit is given by $\mathcal{C}'_0 = \lim_n g'_{i,n}$. Thus taking $n \to \infty$ in (3.22) proves (3.19).

For the final claim, suppose that $w_i \in L^\infty(\mathbb{T}^d)$, $i = 1, 2, 3$, are arbitrary. For each w_i, we can choose a representative such that $|w_i(k)| \le \|w_i\|_\infty$ for *every* k. Then Lusin's theorem implies that there are sequences $v_{i,n} \in C(\mathbb{T}^d)$ such that $|v_{i,n}| \le \|w_i\|_\infty$ and $v_{i,n}(k) \to w_i(k)$ almost everywhere. By dominated convergence, then $v_{i,n} \to w_i$ in L^2. Thus we can define $\mathcal{C}_0[w] \in L^\infty(\mathbb{T}^d)$, $w = (w_1, w_2, w_3)$, by using the sequences $(v_{i,n})$ in the above. Suppose $w_{i,n}$ is some other sequence as in the Lemma, and let $\mathcal{C}_0$ denote the corresponding limit. By (3.19), then $\|\mathcal{C}_0 - \mathcal{C}_0[w]\|_{L^2} = 0$, and hence $\mathcal{C}_0(k) = \mathcal{C}_0[w](k)$ almost everywhere. Therefore, $\mathcal{C}_0[w]$ does not depend on the choice of the approximating sequence. In particular, then for continuous functions (3.20) holds, and proving linearity and commutation with conjugation is straightforward. This finishes the proof of the Lemma. $\square$

Corollary 3.4 *Assume that ω satisfies (DR1) and (DR2) and define $\mathcal{C}_0$ as in Lemma 3.3. For any $W \in L^2_{ferm}$ and $i, i' \in \{1, 2\}$ we define $\mathcal{C}_{\rm diss}[W]_{ii'} \in L^\infty(\mathbb{T}^d)$ by using linearity and $\mathcal{C}_0$: we set*

$$\begin{aligned}
\mathcal{C}_{\rm diss}[W]_{ii'} = \pi \sum_{j\in\{\pm1\}^4} \Big[& \tilde{W}_{ij_1} \left(\delta_{j_2 i'} \mathcal{C}_0[\tilde{W}_{j_3j_4}, W_{j_1j_2}, W_{j_4j_3}] - \delta_{j_3j_4} \mathcal{C}_0[\tilde{W}_{j_2j_3}, W_{j_1j_2}, W_{j_4i'}] \right) \\
& + \tilde{W}_{j_1i'} \left(\delta_{ij_2} \mathcal{C}_0[\tilde{W}_{j_4j_3}, W_{j_2j_1}, W_{j_3j_4}] - \delta_{j_4j_3} \mathcal{C}_0[\tilde{W}_{j_3j_2}, W_{j_2j_1}, W_{ij_4}] \right) \\
& - W_{ij_1} \left(\delta_{j_2i'} \mathcal{C}_0[W_{j_3j_4}, \tilde{W}_{j_1j_2}, \tilde{W}_{j_4j_3}] - \delta_{j_3j_4} \mathcal{C}_0[W_{j_2j_3}, \tilde{W}_{j_1j_2}, \tilde{W}_{j_4i'}] \right) \\
& - W_{j_1i'} \left(\delta_{ij_2} \mathcal{C}_0[W_{j_4j_3}, \tilde{W}_{j_2j_1}, \tilde{W}_{j_3j_4}] - \delta_{j_4j_3} \mathcal{C}_0[W_{j_3j_2}, \tilde{W}_{j_2j_1}, \tilde{W}_{ij_4}] \right) \Big] ,
\end{aligned} \tag{3.23}$$

where $\delta_{ab} := \mathbb{1}(a = b)$ denotes the Kronecker delta. Collecting the components defines a matrix function $\mathcal{C}_{\rm diss}[W] : L^2_{ferm} \to L^2_{\mathbb{H}}$ such that $\mathcal{C}_{\rm diss}[\tilde{W}] = -\mathcal{C}_{\rm diss}[W]$. If $W \in X_{\mathbb{H}}$, then $\mathcal{C}_{\rm diss}[W] \in X_{\mathbb{H}}$ and (1.3) holds pointwise. In addition, there is a constant C, which depends only on ω, such that for all $W, W' \in L^2_{ferm}$,

$$\|\mathcal{C}_{\rm diss}[W]\|_\infty \le C \,, \tag{3.24}$$

$$\|\mathcal{C}_{\rm diss}[W] - \mathcal{C}_{\rm diss}[W']\|_2 \le C \|W' - W\|_2 \,. \tag{3.25}$$

Proof: If M is a matrix such that $0 \le M \le 1$, then $|M_{ij}| \le 1$ for all indices i, j. Since then $0 \le 1 - M \le 1$, we also have $|\tilde{M}_{ij}| \le 1$. Therefore, if $W \in L^2_{\rm ferm}$, then $\tilde{W} \in L^2_{\rm ferm}$, and $\|W_{ij}\|_\infty \le 1$ and $\|\tilde{W}_{ij}\|_\infty \le 1$ for all indices i, j. Let c_0 denote a constant for which the bounds (3.18) and (3.19) in Lemma 3.3 hold. We can conclude that all of the above $\mathcal{C}_0$-terms are well defined and each has an L^∞-norm bounded by c_0. Multiplying this with W_{ij}

or $\tilde{W}_{ij}$ does not increase the bound, and thus we can conclude that $\|\mathcal{C}_{\rm diss}[W]_{ii'}\|_{L^\infty} \le 64\pi c_0$. Also, it is obvious from the definition that $\mathcal{C}_{\rm diss}[W]^*_{ii'} = \mathcal{C}_{\rm diss}[W]_{i'i}$ and hence we have proven that $\mathcal{C}_{\rm diss}[W] \in L^2_{\mathbb{H}}$ and it satisfies (3.24). The property $\mathcal{C}_{\rm diss}[\tilde{W}] = -\mathcal{C}_{\rm diss}[W]$ is also an immediate consequence of the definition (3.23).

If $W \in X_{\mathbb{H}}$, then each of its component functions is continuous, and then by Lemma 3.3 each action of $\mathcal{C}_0$ in (3.23) is given by an integral over the same Borel measure $\nu_{k_1}(\mathrm{d}^3k')$ and the resulting functions are continuous in k_1. Then the sums can be collected inside the integral, and the integrand expressed in terms of matrix products. After some algebra, this proves that then (1.3) holds for every k_1. Therefore, $\mathcal{C}_{\rm diss}[W](k)$ is everywhere a Hermitian matrix, and we can conclude that also $\mathcal{C}_{\rm diss}[W] \in X_{\mathbb{H}}$.

To prove (3.25), assume that $W, W' \in L^2_{\rm ferm}$ are given. Then $\|\mathcal{C}_{\rm diss}[W] - \mathcal{C}_{\rm diss}[W']\|_2^2 = \sum_{ii'} \|\mathcal{C}_{\rm diss}[W]_{ii'} - \mathcal{C}_{\rm diss}[W']_{ii'}\|^2_{L^2}$. By (3.18) and (3.19), $\|\mathcal{C}_{\rm diss}[W]_{ii'} - \mathcal{C}_{\rm diss}[W']_{ii'}\|_{L^2}$ can be bounded by $64\pi \times 4c_0\|W' - W\|_2$. Hence $\|\mathcal{C}_{\rm diss}[W] - \mathcal{C}_{\rm diss}[W']\|_2 \le 8^3\pi c_0\|W' - W\|_2$. This concludes the proof of the corollary. □

We will later also need another consequence of Proposition 3.1: the level sets of $\tilde{\Omega}$ have then Lebesgue measure zero. Note that this is not true in general without assumption (DR2), even for smooth dispersion relations. Consider for instance ω which coincides with a linear map in some neighborhood of 0: then we have $\tilde{\Omega} = 0$ in some sufficiently small ball around zero.

Corollary 3.5 *Assume that $\omega : \mathbb{T}^d \to \mathbb{R}$ is continuous and satisfies (DR2), consider a fixed $\sigma \in \{-1, 1\}^4$, and define $\tilde{\Omega}$ as in (2.2). Then for any $\alpha \in \mathbb{R}$ the set $S_\alpha := \left\{k \in (\mathbb{T}^d)^3 \,\middle|\, \tilde{\Omega}(k,\sigma) = \alpha\right\}$ is compact and $\int_{S_\alpha} \mathrm{d}^3k = 0$.*

Proof: Fix $\alpha \in \mathbb{R}$, and denote $c_0 := \int_{S_\alpha} \mathrm{d}^3k$. By continuity of $\tilde{\Omega}$, S_α is compact, and hence Borel measurable. By Fubini's theorem,

$$\begin{aligned} &\int_{\mathbb{T}^d} \mathrm{d}k_1 \left(\int_{(\mathbb{T}^d)^2} \frac{\mathrm{d}k_2 \mathrm{d}k_3}{\pi} \frac{\varepsilon}{\varepsilon^2 + (\tilde{\Omega}(k,\sigma) - \alpha)^2} \right) \\ &\quad = \int_{(\mathbb{T}^d)^3} \frac{\mathrm{d}^3k}{\pi} \frac{\varepsilon}{\varepsilon^2 + (\tilde{\Omega}(k,\sigma) - \alpha)^2} \ge \int_{S_\alpha} \frac{\mathrm{d}^3k}{\pi} \frac{\varepsilon}{\varepsilon^2 + (\tilde{\Omega}(k,\sigma) - \alpha)^2} = \frac{c_0}{\pi\varepsilon}, \end{aligned} \tag{3.26}$$

for any $\varepsilon > 0$. However, Proposition 3.1 implies that the left hand side converges to a finite value as $\varepsilon \to 0^+$, which is possible only if $c_0 = 0$. □

4 The regularized initial value problem

In this section, we investigate the solutions to the regularized evolution equation. These solutions will provide a sequence of approximations used in the proof of the main theorem. The goal is to prove that the regularized problem is well-posed and preserves continuity in k; in fact, this can be proven even without the assumption (DR3).

The regularization is defined by choosing an arbitrary $\varepsilon > 0$, and setting $\mathcal{C}^\varepsilon[W] := \mathcal{C}_{\text{diss}}[W] + \mathcal{C}^\varepsilon_{\text{cons}}[W]$ where $\mathcal{C}^\varepsilon_{\text{cons}}[W] := -\mathrm{i}[H^\varepsilon_{\text{eff}}[W], W]$ and $H^\varepsilon_{\text{eff}}[W](k_1)$ is defined using the integral in (2.11). By Corollary 3.4, for a given $W \in X_{\mathbb{H}}$ also $\mathcal{C}_{\text{diss}}[W] \in X_{\mathbb{H}}$ and it satisfies (1.3). It is also straightforward to check that $H^\varepsilon_{\text{eff}}[W] \in X_{\mathbb{H}}$ for any $W \in X_{\mathbb{H}}$, and hence also $\mathcal{C}^\varepsilon_{\text{cons}}[W] \in X_{\mathbb{H}}$. Therefore, the regularized collision operator is a well-defined map from $X_{\mathbb{H}}$ to itself and we can hope to solve the regularized evolution problem in the space $X_{\mathbb{H}}$. In fact, we can show that not only is the regularized problem with fermionic initial data well-posed, but it also preserves the Fermi property and the conservation laws.

Theorem 4.1 *Suppose ω satisfies (DR1) and (DR2). If $W_0 \in X_{ferm}$ and $\varepsilon > 0$, then there is a unique $W \in C^{(1)}([0,\infty), X_{ferm})$ such that $W(0,k) = W_0(k)$ for every k, and for all $t > 0$ and every $k \in \mathbb{T}^d$,*

$$\partial_t W_t(k) = \mathcal{C}_{\text{diss}}[W_t](k) - \mathrm{i}[H^\varepsilon_{\text{eff}}[W_t](k), W_t(k)]\,, \tag{4.1}$$

where $W_t(k) := W(t,k)$. In addition, W_t depends continuously on W_0 on any compact interval of $[0,\infty)$, and the solution conserves total energy and spin: equalities (2.14) and (2.15) hold for all $t \geq 0$.

Moreover, to every $k \in \mathbb{T}^d$ and $0 \leq s \leq t$ we can then attach a unitary matrix $U^\varepsilon_{t,s}(k; W_0)$ such that $U^\varepsilon_{t,t}(k; W_0) = 1$ and for which the map $s \mapsto U^\varepsilon_{t,s}(k; W_0)$ belongs to $C^{(1)}([0,t], \mathbb{C}^{2\times 2})$ with

$$\partial_s U^\varepsilon_{t,s}(k; W_0) = \mathrm{i} U^\varepsilon_{t,s}(k; W_0) H^\varepsilon_{\text{eff}}[W_s](k)\,. \tag{4.2}$$

Then also

$$W_t(k) = U^\varepsilon_{t,0}(k; W_0) W_0(k) U^\varepsilon_{t,0}(k; W_0)^* + \int_0^t \mathrm{d}s\, U^\varepsilon_{t,s}(k; W_0) \mathcal{C}_{\text{diss}}[W_s](k) U^\varepsilon_{t,s}(k; W_0)^*\,. \tag{4.3}$$

Proof: Suppose ω satisfies (DR1) and (DR2), and consider a fixed $\varepsilon > 0$. As explained above, then for any $W \in X_{\mathbb{H}}$, both $\mathcal{C}_{\text{diss}}[W]$ and $H^\varepsilon_{\text{eff}}[W]$ are defined directly as integrals over the appropriate measures, and we can use matrix-algebraic manipulations in the integrands to simplify the formulae. First, we observe that in both cases the highest order monomial terms cancel out inside the integrand. For $H^\varepsilon_{\text{eff}}$, we can so find the following alternative, slightly less symmetric but shorter, expression

$$\begin{aligned} H^\varepsilon_{\text{eff}}[W](k_1) = \frac{1}{2} \int_{(\mathbb{T}^d)^3} & \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4\, \delta(\underline{k})\, \frac{\underline{\omega}}{\underline{\omega}^2 + \varepsilon^2} \\ & \times \Big(J[W_4 - W_2] W_3 + W_3 J[W_4 - W_2] + J[W_2 \tilde{W}_4 + \tilde{W}_4 W_2] \Big)\,. \end{aligned} \tag{4.4}$$

Furthermore, we can now split the dissipative part into sum of a "gain" and a "loss" term. Defining

$$\mathcal{G}[W](k_1) := \pi \int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4\, \delta(\underline{k}) \delta(\underline{\omega}) \Big(W_3 J[\tilde{W}_2 W_4] + J[W_4 \tilde{W}_2] W_3 \Big)\,, \tag{4.5}$$

$$\mathcal{D}[W](k_1) := \pi \int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4\, \delta(\underline{k}) \delta(\underline{\omega}) \Big(J[W_4 \tilde{W}_2] W_3 + J[\tilde{W}_4 W_2] \tilde{W}_3 \Big)\,, \tag{4.6}$$

allows to rewrite (1.3) as

$$\mathcal{C}_{\mathrm{diss}}[W](k_1) = \mathcal{G}[W](k_1) - \mathcal{D}[W](k_1)W(k_1) - W(k_1)\mathcal{D}[W](k_1)^* \,. \tag{4.7}$$

Here the first term is called the *gain term* and the rest, the *loss term.*

As mentioned in the beginning of this section, under the present assumptions, $\mathcal{C}^\varepsilon$ maps $X_{\mathbb{H}}$ into itself. There is also an additional symmetry: if W is a solution to $\partial_t W_t = \mathcal{C}^\varepsilon[W_t]$ with initial data W_0, then $\tilde{W}$ is also a solution, with initial data $\tilde{W}_0$. To see this, note that then $\partial_t \tilde{W}_t = -\mathcal{C}^\varepsilon[W_t]$ where $-\mathcal{C}^\varepsilon[W_t] = \mathcal{C}^\varepsilon[\tilde{W}_t]$, as can be seen by using the property $(\tilde{W})\tilde{} = W$ in the integral representations (1.3) and in (2.11): this shows that $\mathcal{C}_{\mathrm{diss}}[\tilde{W}] = -\mathcal{C}_{\mathrm{diss}}[W]$ and $H^\varepsilon_{\mathrm{eff}}[\tilde{W}] = H^\varepsilon_{\mathrm{eff}}[W]$, and hence also that $\mathcal{C}^\varepsilon_{\mathrm{cons}}[\tilde{W}] = -\mathcal{C}^\varepsilon_{\mathrm{cons}}[W]$.

In the proof of the Proposition we follow the strategy used by Dolbeault in [4], albeit with a somewhat different truncation procedure. We begin by introducing a truncation to the collision operator which will be employed to ensure the existence of global solutions. For this, we first set

$$\Phi(x) := \begin{cases} 0, & \text{if } x < 0\,, \\ x, & \text{if } 0 \le x \le 1\,, \\ 1, & \text{if } x > 1\,. \end{cases} \tag{4.8}$$

For any Hermitian matrix M, we then define a positive matrix $\Phi[M]$ via symbolic calculus. Explicitly, let $\lambda_i \in \sigma(M)$, with i counting the eigenvalues, and denote by P_i the corresponding spectral projection operators (which have a two-dimensional range, in case of a degenerate eigenvalue). Then $M = \sum_i \lambda_i P_i$ is the spectral decomposition of M and we define

$$\Phi[M] := \sum_i \Phi(\lambda_i) P_i \,. \tag{4.9}$$

This results in a Lipschitz map, as the following Lemma shows.

Lemma 4.2 *There is a constant $C > 0$ such that $\|\Phi[M'] - \Phi[M]\| \le C\|M' - M\|$ for all $M, M' \in \mathbb{C}^{2\times 2}$ which are Hermitian. In addition, then $1-\Phi[M] = \Phi[1-M]$, $0 \le \Phi[M] \le 1$ and $\Phi[M] = M$ if $0 \le M \le 1$.*

Proof: We first observe that Φ satisfies

$$\Phi(x) = \frac{1}{2}(1 + |x| - |1-x|)\,, \quad x \in \mathbb{R}\,. \tag{4.10}$$

Let $|M|$ denote the absolute value of a matrix M, defined by $|M| := (M^*M)^{\frac{1}{2}}$. Using the above spectral decomposition of M, we find that $|M| = \sum_i |\lambda_i| P_i$, and hence by (4.10)

$$\Phi[M] = \frac{1}{2}(1 + |M| - |1-M|)\,. \tag{4.11}$$

The map $M \to |M|$ is Lipschitz continuous in the Hilbert-Schmidt norm (this is proven, for instance, in [14, Theorem 1]). Hence (4.11) implies that $M \mapsto \Phi[M]$ is also Lipschitz continuous, and there is a pure constant C such that $\|\Phi[M'] - \Phi[M]\| \le C\|M' - M\|$ for all M, M'.

By (4.10), clearly $\Phi(1-x) = 1 - \Phi(x)$ for all $x \in \mathbb{R}$. Since we have $\sum_i P_i = 1$ in the spectral decomposition, it follows that $1 - \Phi[M] = \Phi[1-M]$. The other properties of $\Phi[M]$ listed in the Lemma are obvious consequences of the definition (4.9). $\square$

As the first step in the proof, we will show that for any $w_0 \in X_{\mathbb{H}}$ there is a unique global solution $w \in C^{(1)}([0,\infty), X_{\mathbb{H}})$ which solves the partially truncated evolution equation

$$\partial_t w_t(k) = \mathcal{C}^{\varepsilon}_{\rm tr}[w_t](k)\,, \qquad \mathcal{C}^{\varepsilon}_{\rm tr}[w] := \mathcal{C}_{\rm diss,tr}[w] + \mathcal{C}^{\varepsilon}_{\rm cons,tr}[w]\,, \tag{4.12}$$

where

$$\mathcal{C}_{\rm diss,tr}[w](k) := \mathcal{G}[\Phi[w]](k) - \mathcal{D}[\Phi[w]](k)w(k) - w(k)\mathcal{D}[\Phi[w]](k)^*\,, \tag{4.13}$$

$$\mathcal{C}^{\varepsilon}_{\rm cons,tr}[w](k) := -\mathrm{i}[H^{\varepsilon}_{\rm eff}[\Phi[w]](k), w(k)]\,. \tag{4.14}$$

To prove this, we rely on the standard fixed point methods in the Banach spaces $Y_{t_0} := C([0,t_0], X_{\mathbb{H}})$, $t_0 > 0$, equipped with the norms $\|w\|_Y := \sup_{t,k} |w_t(k)| = \sup_t \|w_t\|$. Given $W_0 \in X_{\mathbb{H}}$ we define for all $w \in Y_{t_0}$, $k \in \mathbb{T}^d$, $0 \le t \le t_0$,

$$\mathbf{T}[w]_t(k) := W_0(k) + \int_0^t \mathrm{d}s\, \mathcal{C}^{\varepsilon}_{\rm tr}[w_s](k)\,. \tag{4.15}$$

We begin by proving that for any $W_0 \in X_{\mathbb{H}}$, there is a non-increasing function $\theta_0(\|W_0\|) > 0$ such that $\mathbf{T}$ is a contractive mapping on the closed ball $\overline{B}(W_0, 1)$ of Y_{θ_0}, where with a slight abuse of notation we have denoted by W_0 also its time-constant extension, i.e., the function $F \in Y_{\theta_0}$ for which $F_t(k) := W_0(k)$ for all t, k.

Suppose $w \in X_{\mathbb{H}}$ and $k \in \mathbb{T}^d$ are arbitrary. Since $\|J[M]\| \le \|M\|$ for any $M \in \mathbb{C}^{2\times 2}$, we have $\|w_1 J[w_2 w_3]\| \le \prod_{i=1}^3 \|w_i\|$ for any choice of $w_i \in \mathbb{C}^{2\times 2}$, $i = 1,2,3$. Together with Proposition 3.1, this implies that there is a constant C_0, depending only on ω, such that

$$\|\mathcal{G}[w](k)\|,\ \|\mathcal{D}[w](k)\| \le C_0(1 + \|w\|)^3\,. \tag{4.16}$$

Therefore, $\|\mathcal{C}_{\rm diss,tr}[w](k)\| \le C_0(1 + \|\Phi[w]\|)^3(1 + 2\|w(k)\|)$, and since $\|\Phi[w]\| \le 2$, we can find a constant C_1, also depending only on ω, such that

$$\|\mathcal{C}_{\rm diss,tr}[w](k)\| \le C_1(1 + \|w(k)\|)\,. \tag{4.17}$$

Similarly, we obtain that there is a constant C_2^{ε}, depending only on ε and ω, such that $\|H^{\varepsilon}_{\rm eff}[\Phi[w]](k)\| \le C_2^{\varepsilon}/2$, and hence $\|\mathcal{C}^{\varepsilon}_{\rm cons,tr}[w](k)\| \le C_2^{\varepsilon}\|w(k)\|$. Thus

$$\|\mathcal{C}^{\varepsilon}_{\rm tr}[w](k)\| \le R(1 + \|w(k)\|)\,, \tag{4.18}$$

where $R := C_1 + C_2^{\varepsilon}$ depends only on ε and ω.

Suppose $w \in X_{\mathbb{H}}$. By item 4 of Proposition 3.1, then both $\mathcal{G}[w]$ and $\mathcal{D}[w]$ are continuous functions in k. On the other hand, for instance by using dominated convergence, we find that also $k \mapsto H^{\varepsilon}_{\text{eff}}[w](k)$ is continuous. By Lemma 4.2, the map $M \mapsto \Phi[M]$ is continuous in the matrix norm, and thus we can conclude that for any $w \in X_{\mathbb{H}}$ we have $\mathcal{C}^{\varepsilon}_{\text{tr}}[w] \in X_{\mathbb{H}}$ with

$$\|\mathcal{C}^{\varepsilon}_{\text{tr}}[w]\| \leq R(1 + \|w\|) \,. \tag{4.19}$$

Note that $\mathcal{G}[\Phi[w]](k)$ and $H^{\varepsilon}_{\text{eff}}[\Phi[w]](k)$ are clearly Hermitian for $w \in X_{\mathbb{H}}$, $k \in \mathbb{T}^d$, and thus so is $\mathcal{C}^{\varepsilon}_{\text{tr}}[w](k)$.

Let us next consider the continuity properties of the map $w \mapsto \mathcal{C}^{\varepsilon}_{\text{tr}}[w]$ in $X_{\mathbb{H}}$. The following Lemma generalizing a result of Seiler and Simon [15] will become useful for this purpose. The corresponding statement and a proof for scalar valued functions is given in [16, Theorem 4.1]. In fact, the proof carries over verbatim for matrix valued functions, as soon as one understand all integrals as "vector valued" in the sense of used in topological vector spaces [13]. The full proof is included here mainly for the sake of completeness.

Lemma 4.3 (Lipschitz bounds) *Let f be a complex matrix valued function defined on complex normed linear space $\mathcal{N}$. Suppose that*

1. *the function $\lambda \mapsto f(A + \lambda B)$ is an entire matrix function for all A, B in $\mathcal{N}$, and*
2. *there is monotone non-decreasing function g on $[0, \infty)$ such that for all $A \in \mathcal{N}$,*

$$\|f(A)\| \leq g(\|A\|_{\mathcal{N}}) \,. \tag{4.20}$$

Then for all A, B in $\mathcal{N}$,

$$\|f(A) - f(B)\| \leq \|A - B\|_{\mathcal{N}} \, g(\|A\|_{\mathcal{N}} + \|B\|_{\mathcal{N}} + 1) \,. \tag{4.21}$$

Proof: Let $h(\lambda) := f(\frac{1}{2}(A + B) + \lambda(A - B))$. By assumption, then h is an entire matrix function and

$$\|f(A) - f(B)\| = \left\| h\left(\frac{1}{2}\right) - h\left(-\frac{1}{2}\right) \right\| \leq \sup_{-\frac{1}{2} \leq t \leq \frac{1}{2}} \|h'(t)\| \,. \tag{4.22}$$

By assumption, the matrix valued map h is holomorphic on $\mathbb{C}$. From the Cauchy contour integral formula (see for instance the proof of Theorem 3.31 in [13] for details) we get for $\rho > 0$ and $-\frac{1}{2} \leq t \leq \frac{1}{2}$

$$\|h'(t)\| = \frac{1}{2\pi} \left\| \oint_{|s|=\rho} \frac{h(s+t)}{s^2} \mathrm{d}s \right\| \leq \frac{1}{\rho} \sup_{|s|=\rho} \|h(s+t)\| \leq \frac{1}{\rho} \sup_{|\lambda| \leq \rho + \frac{1}{2}} \|h(\lambda)\| \,. \tag{4.23}$$

Therefore, for any $\rho > 0$,

$$\|f(A) - f(B)\| \leq \frac{1}{\rho} \sup_{|\lambda| \leq \rho + \frac{1}{2}} \|h(\lambda)\| \,. \tag{4.24}$$

If $A = B$, then (4.21) obviously holds. Assume thus $A \neq B$, and set $\rho := \|A - B\|_{\mathcal{N}}^{-1}$. Then for any $|\lambda| \le \rho + \frac{1}{2}$ we have

$$\left\|\frac{1}{2}(A+B) + \lambda(A-B)\right\|_{\mathcal{N}} \le \frac{1}{2}(\|A+B\|_{\mathcal{N}} + \|A-B\|_{\mathcal{N}}) + \rho\,\|A-B\|_{\mathcal{N}}$$
$$\le \|A\|_{\mathcal{N}} + \|B\|_{\mathcal{N}} + 1\,. \tag{4.25}$$

Hence, if $|\lambda| \le \rho + \frac{1}{2}$, then by the second assumption we can estimate

$$\|h(\lambda)\| = \left\|f\left(\frac{1}{2}(A+B) + \lambda(A-B)\right)\right\| \le g\left(\left\|\frac{1}{2}(A+B) + \lambda(A-B)\right\|_{\mathcal{N}}\right)\,, \tag{4.26}$$

where, by the monotonicity of g, the right hand side is bounded by $g(\|A\|_{\mathcal{N}} + \|B\|_{\mathcal{N}} + 1)$. Therefore, (4.24) implies now that also for $A \neq B$ (4.21) holds. □

Consider then some $k \in \mathbb{T}^d$. Since $\|\mathcal{G}[w](k)\| \le C_0(1 + \|w\|)^3$, the map $w \mapsto \mathcal{G}[w](k)$ satisfies the conditions[2] of the Lemma with $g(x) := C_0(1+x)^3$. Hence, for all $w, w' \in X_{\mathbb{H}}$ and any $k \in \mathbb{T}^d$ we have $\|\mathcal{G}[w'](k) - \mathcal{G}[w](k)\| \le \|w' - w\| C_0(\|w'\| + \|w\| + 2)^3$. Therefore, by Lemma 4.2, there is a constant c_0', which depends only on ω, such that $\|\mathcal{G}[\Phi[w']] - \mathcal{G}[\Phi[w]]\| \le c_0'\|w' - w\|$. Thus $w \mapsto \mathcal{G}[\Phi[w]]$ is a Lipschitz map on $X_{\mathbb{H}}$. Completely analogous reasoning shows that also $w \mapsto \mathcal{D}[\Phi[w]]$ and $w \mapsto H^{\varepsilon}_{\text{eff}}[\Phi[w]]$ are Lipschitz maps $X_{\mathbb{H}} \to C(\mathbb{T}^d, \mathbb{C}^{2\times 2})$, with Lipschitz constants which depend only on ω and ε. Therefore, using also the earlier derived uniform bounds, we can conclude that there is a constant R', which depends only on ω and ε, such that

$$\|\mathcal{C}^{\varepsilon}_{\text{tr}}[w'] - \mathcal{C}^{\varepsilon}_{\text{tr}}[w]\| \le R'(1 + \|w\| + \|w'\|)\|w' - w\|\,, \qquad w, w' \in X_{\mathbb{H}}\,. \tag{4.27}$$

Consider then an arbitrary $t_0 > 0$. If $w \in Y_{t_0}$, then dominated convergence and the above bounds imply that the function $\mathbf{T}[w]$ defined by (4.15) is continuous, both in t and in k, and $\mathbf{T}[w]_t(k)$ is always Hermitian and satisfies the bound

$$\|\mathbf{T}[w]_t(k) - W_0(k)\| \le R\int_0^t \mathrm{d}s\,(1 + \|w_s\|)\,, \quad w \in Y_{t_0}\,. \tag{4.28}$$

In addition, by (4.27) for any $w', w \in Y_{t_0}$,

$$\|\mathbf{T}[w']_t(k) - \mathbf{T}[w]_t(k)\| \le tR'(1 + \|w\|_Y + \|w'\|_Y)\|w' - w\|_Y\,, \quad t \in [0, t_0],\ k \in \mathbb{T}^d\,. \tag{4.29}$$

Set $\theta_0(\|W_0\|) := c_2/(1 + \|W_0\|)$ where $c_2 := \min((8R')^{-1}, (2R)^{-1})$ depends only on ω and ε, and thus θ_0 is a non-increasing function of $\|W_0\|$ with $\theta_0 > 0$ for all $\|W_0\| < \infty$. Then, if $t_0 \le \theta_0$, the above estimates imply $\|\mathbf{T}[w] - W_0\|_Y \le 1$ and $\|\mathbf{T}[w']_t(k) - \mathbf{T}[w]_t(k)\|_Y \le$

[2] In principle, we have defined the map only in the real Banach space $X_{\mathbb{H}}$. However, it is obvious that the defining integral can also be applied in the complex Banach space $C(\mathbb{T}^d, \mathbb{C}^{2\times 2})$ and that none of the bounds used the fact that $w(k)$ is Hermitian.

$\|w' - w\|_Y/2$ whenever $w', w \in Y_{t_0}$ satisfy $\|w' - W_0\|_Y, \|w - W_0\|_Y \le 1$. This proves the earlier claim that for any $0 < t_0 \le \theta_0$ the map $\mathbf{T}$ is a contraction from the closed ball $\overline{B}(W_0, 1)$ into itself. Thus we can conclude from the Banach fixed point theorem that for any $W_0 \in X_{\mathbb{H}}$ there is a unique $w \in \overline{B}(W_0, 1) \subset Y_{\theta_0}$, $\theta_0 = \theta_0(\|W_0\|) > 0$, for which

$$w_t(k) = W_0(k) + \int_0^t \mathrm{d}s\, \mathcal{C}^{\varepsilon}_{\mathrm{tr}}[w_s](k)\,, \quad 0 \le t \le \theta_0,\ k \in \mathbb{T}^d\,. \tag{4.30}$$

Fix then $W_0 \in X_{\mathbb{H}}$ and consider the collection of solutions to (4.30), i.e., the set of (t_0, w), with $t_0 > 0$, $w \in Y_{t_0}$, such that (4.30) holds for $0 \le t \le t_0$. By the above result we know that this set is not empty, and extension of functions clearly defines a partial order on it. By Hausdorff's maximality principle, there is a maximal totally ordered subset. Let T_0 denote the supremum of the t_0 in this set, and define $w_t(k)$ for $0 \le t < T_0$, $k \in \mathbb{T}^d$, by a choosing a function from the set with $t_0 > t$ (such a t_0 must exist and the value of $w_t(k)$ does not depend on the choice since the set is totally ordered). Then $w \in C([0, T_0), X_{\mathbb{H}})$, and it is a maximal solution to (4.30) for $0 \le t < T_0$. Now (4.19) and (4.30) imply

$$\|w_t(k)\| \le \|W_0\| + \int_0^t \mathrm{d}s\, R(1 + \|w_s\|)\,, \quad 0 \le t < T_0\,,\ k \in \mathbb{T}^d\,. \tag{4.31}$$

The map $s \mapsto \|w_s\|$ is continuous, and thus Grönwall's lemma can be applied on the time-interval $[0, t]$. This allows to conclude that

$$\|w_t\| \le (\|W_0\| + Rt)\mathrm{e}^{Rt}\,, \quad 0 \le t < T_0\,. \tag{4.32}$$

If we suppose that $T_0 < \infty$, this would imply that $\|w_t\| \le c := (\|W_0\| + RT_0)\mathrm{e}^{RT_0}$ for all $0 \le t < T_0$. However, if we then apply the fixed point result to initial data $w_{T_0 - \theta_0(c)/2}$ we obtain an extension up to times $T_0 + \theta_0(c)/2$. This contradicts the maximality of w, and hence necessarily $T_0 = \infty$.

We have now proven that for any $W_0 \in X_{\mathbb{H}}$ there is $w \in C([0, \infty), X_{\mathbb{H}})$ which satisfies (4.30) for all t, k. Since $\mathcal{C}^{\varepsilon}_{\mathrm{tr}}[w_t](k)$ is continuous, this directly implies that $w \in C^{(1)}([0, \infty), X_{\mathbb{H}})$ with $\partial_t w_t = \mathcal{C}^{\varepsilon}_{\mathrm{tr}}[w_t]$, for all $t \ge 0$, and that $w_t \to W_0$ as $t \to 0^+$. Thus it provides a global solution to the truncated Cauchy problem. Suppose v is another global solution corresponding to some initial data $V_0 \in X_{\mathbb{H}}$. Then it also satisfies (4.30) for all t, k and hence also (4.32). Therefore, by (4.27)

$$\begin{aligned}
\|v_t(k) - w_t(k)\| &\le \|V_0 - W_0\| + \int_0^t \mathrm{d}s\, \|\mathcal{C}^{\varepsilon}_{\mathrm{tr}}[v_s] - \mathcal{C}^{\varepsilon}_{\mathrm{tr}}[w_s]\| \\
&\le \|V_0 - W_0\| + R'(1 + \|V_0\| + \|W_0\| + 2Rt)\mathrm{e}^{Rt} \int_0^t \mathrm{d}s\, \|v_s - w_s\|\,,
\end{aligned} \tag{4.33}$$

for all $t \ge 0$ and $k \in \mathbb{T}^d$. Thus Grönwall's lemma implies that, if $t \in [0, t_0]$, with $t_0 > 0$ arbitrary, then

$$\|v_t - w_t\| \le \|V_0 - W_0\|\mathrm{e}^{t_0 R'(1 + \|V_0\| + \|W_0\| + 2Rt_0)}\,. \tag{4.34}$$

This shows that the global solution is unique, and also proves that it depends continuously on the initial data.

We have now proven that the truncated problem (4.12) is well-posed for any initial data $W_0 \in X_{\mathbb{H}}$. Our next goal is to show that such solutions preserve the Fermi property, i.e., to show that if $W_0 \in X_{\rm ferm}$, then the corresponding solution w satisfies $w_t \in X_{\rm ferm}$ for all $t \geq 0$. Suppose this is the case. Then $\Phi[w_t] = w_t$, and thus $\mathcal{C}^\varepsilon_{\rm tr}[w_t] = \mathcal{C}^\varepsilon[w_t]$ for all t. It follows that then $w \in C^{(1)}([0,\infty), X_{\rm ferm})$, $w_0 = W_0$, and $\partial_t w_t = \mathcal{C}^\varepsilon[w_t]$ for all $t > 0$; therefore, choosing $W = w$ yields a solution satisfying the conditions of the Theorem. It is also the only such function: for any W as in the Theorem, we have $W_t \in X_{\rm ferm}$ implying $\mathcal{C}^\varepsilon[W_t] = \mathcal{C}^\varepsilon_{\rm tr}[W_t]$, and thus W then is a solution to the truncated problem, hence equal to w. Finally, (4.34) then immediately implies that the unique solution depends continuously on the initial data.

Therefore, to complete the proof of the first part of the Theorem we only need to show that the above solutions preserve the Fermi property and the conservation laws. In fact, for the Fermi property it suffices to show that the solutions preserve positivity in the matrix sense. Indeed, suppose that we have proven that for every $W_0 \in X_{\mathbb{H}}$ with $W_0 \geq 0$ necessarily $w_t \geq 0$ for all t. We will soon show that also the truncated problem preserves the $W \to \tilde{W}$ symmetry, i.e., we will show that if w is a solution to $\partial_t w_t = \mathcal{C}^\varepsilon_{\rm tr}[w_t]$ with initial data W_0, then $\tilde{w}$ is a solution with initial data $\tilde{W}_0$. If $W_0 \in X_{\rm ferm}$, we have $W_0 \geq 0$ and $\tilde{W}_0 \geq 0$, and as positivity of solutions is preserved, we can then conclude that $w_t \geq 0$ and $\tilde{w}_t \geq 0$, and thus $0 \leq w_t \leq 1$.

To prove the symmetry statement suppose $W_0 \in X_{\mathbb{H}}$ and let w denote the corresponding solution. Then $\partial_t \tilde{w}_t = -\mathcal{C}^\varepsilon_{\rm tr}[w_t]$, and since $\tilde{w}_0 = \tilde{W}_0$ it suffices to show that $\mathcal{C}^\varepsilon_{\rm tr}[\tilde{v}] = -\mathcal{C}^\varepsilon_{\rm tr}[v]$ for all $v \in X_{\mathbb{H}}$. For this, first note that by Lemma 4.2, we have always $\Phi[\tilde{v}] = (\Phi[v])\tilde{}$. Hence, by the earlier discussion $\mathcal{C}^\varepsilon[\Phi[\tilde{v}]] = -\mathcal{C}^\varepsilon[\Phi[v]]$ and $H^\varepsilon_{\rm eff}[\Phi[\tilde{v}]] = H^\varepsilon_{\rm eff}[\Phi[v]]$. Therefore, now $\mathcal{C}^\varepsilon_{\rm cons,tr}[\tilde{v}] = -\mathcal{C}^\varepsilon_{\rm cons,tr}[v]$ and also $\mathcal{D}[\Phi[\tilde{v}]] = \mathcal{D}[\Phi[v]]$, since it is obvious from the definition (4.6) that $\mathcal{D}[\tilde{W}] = \mathcal{D}[W]$. Employing the above equalities shows that

$$\mathcal{C}^\varepsilon_{\rm tr}[\tilde{v}] + \mathcal{C}^\varepsilon_{\rm tr}[v] = \mathcal{G}[1-\Phi[v]] + \mathcal{G}[\Phi[v]] - \mathcal{D}[\Phi[v]] - \mathcal{D}[\Phi[v]]^* \,. \tag{4.35}$$

However, for any $W \in X_{\mathbb{H}}$, we find directly from the definitions (4.5) and (4.6) that

$$\begin{aligned}
&\mathcal{G}[\tilde{W}](k_1) + \mathcal{G}[W](k_1) \\
&\quad = \pi \int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \, \delta(\underline{k})\delta(\underline{\omega}) \\
&\qquad \times \left(\tilde{W}_3 J[W_2 \tilde{W}_4] + J[\tilde{W}_4 W_2]\tilde{W}_3 + W_3 J[\tilde{W}_2 W_4] + J[W_4 \tilde{W}_2] W_3 \right) \\
&\quad = \mathcal{D}[W](k_1) + \mathcal{D}[W](k_1)^* \,.
\end{aligned} \tag{4.36}$$

Thus (4.35) implies $\mathcal{C}^\varepsilon_{\rm tr}[\tilde{v}] = -\mathcal{C}^\varepsilon_{\rm tr}[v]$, and proves the stated preservation of the $W \to \tilde{W}$ symmetry.

Therefore, to prove the preservation of the Fermi property, we now only need to show that the solutions preserve positivity. The key ingredient in this proof is Lemma 4.5, which implies that the truncated gain term is always a nonnegative matrix. Its proof will rely on the following matrix in equality.

Lemma 4.4 *For any $n \times n$ matrices $A, B, C \geq 0$, $n \geq 1$, we have*

$$AJ[BC] + CJ[BA] \geq 0 \quad \text{and} \quad J[AB]C + J[CB]A \geq 0\,. \tag{4.37}$$

Proof: By expanding the definition of J, we find that $AJ[BC] + CJ[BA] = A\,\mathrm{tr}(BC) + C\,\mathrm{tr}(AB) - ABC - CBA$. To prove that this is nonnegative, choose a complete set of eigenvectors for each Hermitian matrix: let $(a_i, \alpha_i)_{i=1,\ldots,n}$ be an eigensystem for A, $(b_i, \beta_i)_i$ for B, and $(c_i, \gamma_i)_i$ for C. Then for any $\psi \in \mathbb{C}^n$, by selecting suitable bases to express the matrix products and to compute the traces, we obtain

$$\begin{aligned}
&\langle \psi, (A\,\mathrm{tr}(BC) + C\,\mathrm{tr}(AB) - ABC - CBA)\psi\rangle \\
&\quad = \sum_{i,j,k=1,\ldots,n} a_i b_j c_k \big(\langle \psi, \alpha_i\rangle\langle \alpha_i, \psi\rangle\langle \beta_j, \gamma_k\rangle\langle \gamma_k, \beta_j\rangle + \langle \psi, \gamma_k\rangle\langle \gamma_k, \psi\rangle\langle \beta_j, \alpha_i\rangle\langle \alpha_i, \beta_j\rangle \\
&\qquad - \langle \psi, \alpha_i\rangle\langle \alpha_i, \beta_j\rangle\langle \beta_j, \gamma_k\rangle\langle \gamma_k, \psi\rangle - \langle \psi, \gamma_k\rangle\langle \gamma_k, \beta_j\rangle\langle \beta_j, \alpha_i\rangle\langle \alpha_i, \psi\rangle\big) \\
&\quad = \sum_{i,j,k} a_i b_j c_k \big|\langle \psi, \alpha_i\rangle\langle \beta_j, \gamma_k\rangle - \langle \psi, \gamma_k\rangle\langle \beta_j, \alpha_i\rangle\big|^2 \geq 0\,,
\end{aligned} \tag{4.38}$$

since, by assumption, $a_i, b_i, c_i \geq 0$. This implies that the matrix is non-negative. Taking an adjoint proves then that also $J[AB]C + J[CB]A \geq 0$. □

Lemma 4.5 *If $W \in X_{\mathit{ferm}}$, then $\mathcal{G}[W](k) \geq 0$ for all k. Therefore, $\mathcal{G}[\Phi[W]](k) \geq 0$ for all k and $W \in X_{\mathbb{H}}$.*

Proof: It follows directly from its definition in Proposition 3.1 that the measure ν_{k_1} is invariant under the exchange $k_4 \leftrightarrow k_3$. To see this, one can change the integration variable k_3 to $k_4 := k_1 + k_2 - k_3$ in (3.2) and note that $\tilde{\Omega}((k_1, k_2, k_1 + k_2 - k_4), (1, 1, -1, -1)) = \tilde{\Omega}((k_1, k_2, k_4), (1, 1, -1, -1))$. Using this symmetry in the definition (4.5) shows that

$$\begin{aligned}
\mathcal{G}[W](k_1) = \frac{\pi}{2} \int_{(\mathbb{T}^d)^3} & \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4\, \delta(\underline{k})\delta(\underline{\omega}) \\
& \times \Big(W_3 J[\tilde{W}_2 W_4] + J[W_4 \tilde{W}_2] W_3 + W_4 J[\tilde{W}_2 W_3] + J[W_3 \tilde{W}_2] W_4\Big)\,.
\end{aligned} \tag{4.39}$$

If $W \in X_{\mathrm{ferm}}$, then $\tilde{W}_2, W_3, W_4 \geq 0$ inside the integrand above. Therefore, we can apply Lemma 4.4 and conclude that the integrand is pointwise a positive matrix. This directly implies that $\mathcal{G}[W](k_1) \geq 0$, since then for any $\psi \in \mathbb{C}^2$ clearly $\langle \psi, \mathcal{G}[W](k_1)\psi\rangle \geq 0$.

If $W \in X_{\mathbb{H}}$, then $\Phi[W] \in X_{\mathrm{ferm}}$, and thus the second statement is a corollary of the first one. □

With the above preparations, we are now ready to prove the preservation of positivity. Fix thus some $W_0 \in X_{\mathbb{H}}$ with $W_0 \geq 0$ and let w denote the corresponding global solution to the truncated problem. Set $h_t(k) := H^{\varepsilon}_{\rm eff}[\Phi[w_t]](k)$. Then each $h_t(k)$ is a Hermitian matrix, and since the map $t \mapsto h_t(k)$ is also norm continuous for each k, the standard Dyson expansion techniques (see, e.g., [17, Theorem X.69] and take an adjoint of the result) imply that for any $k \in \mathbb{T}^d$ and s, t, with $0 \le s \le t$, we can find a unitary matrix $u_{t,s}(k)$ such that $u_{t,t}(k) = 1$, the map $s \mapsto u_{t,s}(k)$ belongs to $C^{(1)}([0,t], \mathbb{C}^{2\times 2})$, and

$$\partial_s u_{t,s}(k) = \mathrm{i} u_{t,s}(k) h_s(k)\,. \tag{4.40}$$

Given matrices $u_{t,s}(k)$ as above, let us define

$$v_{t,s}(k) := u_{t,s}(k) w_s(k) u_{t,s}(k)^*\,, \quad 0 \le s \le t\,,\ k \in \mathbb{T}^d\,. \tag{4.41}$$

By (4.12) and (4.41) then for any $t > 0$ and $0 < s < t$

$$\begin{aligned}\partial_s v_{t,s}(k) &= u_{t,s}(k)\left(\mathrm{i} h_s(k) w_s(k) + \partial_s w_s(k) - \mathrm{i} w_s(k) h_s(k)\right) u_{t,s}(k)^* \\ &= u_{t,s}(k) \mathcal{C}_{\rm diss,tr}[w_s](k) u_{t,s}(k)^* \\ &= g_{t,s}(k) - b_{t,s}(k) v_{t,s}(k) - v_{t,s}(k) b_{t,s}(k)^*\,. \end{aligned} \tag{4.42}$$

where on the last step we have used the unitarity of $u_{t,s}(k)$ and introduced the shorthand notations

$$g_{t,s}(k) := u_{t,s}(k) \mathcal{G}[\Phi[w_s]](k) u_{t,s}(k)^*\,, \tag{4.43}$$

$$b_{t,s}(k) := u_{t,s}(k) \mathcal{D}[\Phi[w_s]](k) u_{t,s}(k)^*\,. \tag{4.44}$$

By Lemma 4.5, here $g_{t,s}(k) \ge 0$. Fix for the moment $t > 0$ and $k \in \mathbb{T}^d$. Since the map $s \mapsto b_{t,s}(k)$ belongs to $C([0,t], \mathbb{C}^{2\times 2})$, there is is unique solution $F \in C^{(1)}([0,t], \mathbb{C}^{2\times 2})$, $0 \le s \le t$, to the matrix equation $\partial_s F(s) = -F(s) b_{t,t-s}(k)$, with initial data $F(0) = 1$. (The solution can be obtained by a time-ordered exponential, similarly to $u_{t,s}(k)$; more details about matrix equations of this type can be found for instance in [18].) Then $\partial_s F(s)^* = -b_{t,t-s}(k)^* F(s)^*$ and (4.42) implies that

$$\partial_s (F(t-s) v_{t,s}(k) F(t-s)^*) = F(t-s) g_{t,s}(k) F(t-s)^*\,. \tag{4.45}$$

Since $v_{t,t}(k) = w_t(k)$ and $v_{t,0}(k) = u_{t,0}(k) w_0(k) u_{t,0}(k)^*$, we find that

$$w_t(k) = (F(t) u_{t,0}(k)) W_0(k) (F(t) u_{t,0}(k))^* + \int_0^t \mathrm{d}s\, F(t-s) g_{t,s}(k) F(t-s)^*\,. \tag{4.46}$$

As mentioned above, here $g_{t,s}(k) \ge 0$ for all s and, since by assumption $W_0(k) \ge 0$, the above formula shows that $w_t(k) \ge 0$. Since t and k were arbitrary, we can conclude that $w \ge 0$ if $W_0 \ge 0$.

This shows that the truncated time-evolution preserves positivity which was the missing part from the well-posedness result. Therefore, we can now also conclude that, if $W_0 \in$

X_{ferm}, then $w_t \in X_{\mathrm{ferm}}$ for all t, and $W_t := w_t$ provides a solution to the original evolution equation. Integrating (4.42) over s then yields

$$W_t(k) = u_{t,0}(k)W_0(k)u_{t,0}(k)^* + \int_0^t \mathrm{d}s\, u_{t,s}(k)\mathcal{C}_{\mathrm{diss,tr}}[W_s](k)u_{t,s}(k)^* \,. \tag{4.47}$$

Since $\Phi[W_s] = W_s$, here $\mathcal{C}_{\mathrm{diss,tr}}[W_s](k) = \mathcal{C}_{\mathrm{diss}}[W_s](k)$, and $u_{t,s}(k)$ satisfies $u_{t,t}(k) = 1$ and $\partial_s u_{t,s}(k) = \mathrm{i}u_{t,s}(k)H^{\varepsilon}_{\mathrm{eff}}[W_s](k)$. Therefore, the second paragraph of the Theorem holds with the choice $U^{\varepsilon}_{t,s}(k; W_0) := u_{t,s}(k)$.

It only remains to prove that the above solution also preserves energy and spin. For this, consider some $W_0 \in X_{\mathrm{ferm}}$, and let w denote the corresponding solution satisfying $\partial_t w_t(k_1) = \mathcal{C}^{\varepsilon}[w_t](k_1)$ for all $t > 0$, $k_1 \in \mathbb{T}^d$. Then $w_t(k)$ satisfies (4.30) for all t, k and this, together with the uniform bounds in (4.19), allows using Fubini's theorem in the definitions of the conserved quantities. Therefore, it is sufficient to check that for all $W \in X_{\mathrm{ferm}}$

$$\int_{\mathbb{T}^d} \mathrm{d}k_1\, \omega(k_1)\, \mathrm{tr}\, \mathcal{C}^{\varepsilon}[W](k_1) = 0\,, \qquad \int_{\mathbb{T}^d} \mathrm{d}k_1\, \mathcal{C}^{\varepsilon}[W](k_1) = 0\,. \tag{4.48}$$

Let us begin with the first, scalar valued case, implying conservation of energy. First, by cyclicity of trace, $\mathrm{tr}\, \mathcal{C}^{\varepsilon}[W](k_1) = \mathrm{tr}\, \mathcal{C}_{\mathrm{diss}}[W](k_1)$. Since W is continuous, we can evaluate the integral over ν_{k_1} by using the formula (3.2) where, to avoid confusion, let us denote the new regularizing variable by ε_0 instead of ε. As shown in the proof of Proposition 3.1, the resulting integral is uniformly bounded in k_1 and ε_0, hence dominated convergence can be applied to prove that $\int_{\mathbb{T}^d} \mathrm{d}k_1\, \omega(k_1)\, \mathrm{tr}\, \mathcal{C}^{\varepsilon}[W](k_1)$ is equal to the $\varepsilon_0 \to 0^+$ limit of the integral

$$\begin{aligned} &\int_{(\mathbb{T}^d)^4} \mathrm{d}^4 k\, \delta(\underline{k}) \frac{\varepsilon_0}{\varepsilon_0^2 + \underline{\omega}^2} \omega_1 \\ &\quad \times \mathrm{tr}\left(\tilde{W}_1 W_3 J[\tilde{W}_2 W_4] + J[W_4 \tilde{W}_2] W_3 \tilde{W}_1 - W_1 \tilde{W}_3 J[W_2 \tilde{W}_4] - J[\tilde{W}_4 W_2] \tilde{W}_3 W_1 \right) . \end{aligned} \tag{4.49}$$

The trace-factor on the second line changes sign if we relabel the integration variables by $k_1 \leftrightarrow k_3$ and $k_2 \leftrightarrow k_4$, and it is invariant under the relabelling $k_1 \leftrightarrow k_2$, $k_3 \leftrightarrow k_4$; these properties can be proven by using the cyclicity of the trace and the definition of J in (1.6). As both relabellings leave $|\underline{\omega}|$ and $|\underline{k}|$ invariant, by first taking the average over the first swap and then the average of performing the second swap to the result, we find that the value of (4.49) does not change if we replace the factor ω_1 by $(\omega_1 - \omega_3 + \omega_2 - \omega_4)/4 = \underline{\omega}/4$ there. However, then we can retrace the steps above, and produce an integral over ν_{k_1} which contains a factor $\underline{\omega}$. By item 3 in Proposition 3.1, the value of such an integral is zero. Hence we have shown that the first equality in (4.48) holds.

Let us then consider the second, matrix equality, in (4.48). We use the split $\mathcal{C}^{\varepsilon} = \mathcal{C}_{\mathrm{diss}} + \mathcal{C}^{\varepsilon}_{\mathrm{cons}}$ and show independently that both of the resulting two terms are zero. It is easier to check the conservative term using the following integral representation, analogous to the dissipative term:

$$
\begin{aligned}
&\int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4\, \delta(\underline{k}) \frac{\underline{\omega}}{\underline{\omega}^2+\varepsilon^2} \\
&\qquad \times \Big(\tilde{W}_1 W_3 J[\tilde{W}_2 W_4] - J[W_4 \tilde{W}_2] W_3 \tilde{W}_1 - W_1 \tilde{W}_3 J[W_2 \tilde{W}_4] + J[\tilde{W}_4 W_2] \tilde{W}_3 W_1 \Big) \\
&\quad = - \int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4\, \delta(\underline{k}) \frac{\underline{\omega}}{\underline{\omega}^2+\varepsilon^2} \Big(W_3 \tilde{W}_2 W_4 - W_4 \tilde{W}_2 W_3 \Big) \\
&\qquad + \int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4\, \delta(\underline{k}) \frac{\underline{\omega}}{\underline{\omega}^2+\varepsilon^2} \\
&\qquad \times \Big((J[W_4 \tilde{W}_2] W_3 + J[\tilde{W}_4 W_2] \tilde{W}_3) W_1 - W_1 (W_3 J[\tilde{W}_2 W_4] + \tilde{W}_3 J[W_2 \tilde{W}_4]) \Big) \\
&\quad = [H^\varepsilon_{\mathrm{eff}}[W](k_1), W(k_1)] \\
&\quad = \mathrm{i}\mathcal{C}^\varepsilon_{\mathrm{cons}}[W](k_1)\,,
\end{aligned} \tag{4.50}
$$

where in the second equality we have used the definition (2.11) and the property that any term, whose integrand is antisymmetric under the swap $k_3 \leftrightarrow k_4$, evaluates to zero. Then we integrate the equality over k_1, use Fubini's theorem, and take an average over the result from the swap $k_1 \leftrightarrow k_3$, $k_2 \leftrightarrow k_4$, yielding

$$
\begin{aligned}
&\mathrm{i} \int_{\mathbb{T}^d} \mathrm{d}k_1\, \mathcal{C}^\varepsilon_{\mathrm{cons}}[W](k_1) \\
&\quad = \frac{1}{2} \int_{(\mathbb{T}^d)^4} \mathrm{d}^4 k\, \delta(\underline{k}) \frac{\underline{\omega}}{\underline{\omega}^2+\varepsilon^2} \\
&\quad \times \Big(\tilde{W}_1 W_3 J[\tilde{W}_2 W_4] - J[W_4 \tilde{W}_2] W_3 \tilde{W}_1 - W_1 \tilde{W}_3 J[W_2 \tilde{W}_4] + J[\tilde{W}_4 W_2] \tilde{W}_3 W_1 \\
&\qquad - \tilde{W}_3 W_1 J[\tilde{W}_4 W_2] + J[W_2 \tilde{W}_4] W_1 \tilde{W}_3 + W_3 \tilde{W}_1 J[W_4 \tilde{W}_2] - J[\tilde{W}_2 W_4] \tilde{W}_1 W_3 \Big)\,.
\end{aligned} \tag{4.51}
$$

If we expand the definitions of J in the integrand, all terms containing a trace cancel out. The remaining integrand is antisymmetric under the swap $k_1 \leftrightarrow k_2$, $k_3 \leftrightarrow k_4$, hence evaluates to zero.

We have thus proven that $\int_{\mathbb{T}^d} \mathrm{d}k_1\, \mathcal{C}^\varepsilon_{\mathrm{cons}}[W](k_1) = 0$. The proof of $\int_{\mathbb{T}^d} \mathrm{d}k_1\, \mathcal{C}^\varepsilon_{\mathrm{diss}}[W](k_1) = 0$ follows by first expressing the integral as a limit of terms analogous to (4.49) and then performing the swaps as in (4.51); we skip the details of the computation here. This concludes the proof of the conservation laws, and thus also of the Theorem. □

5 L^2-continuity of the collision operator

In this section, we consider the regularized effective Hamiltonian starting from (4.4),

$$
\begin{aligned}
H^\varepsilon_{\mathrm{eff}}[W](k_1) = \frac{1}{2} \int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4\, \delta(\underline{k})\, \frac{\underline{\omega}}{\underline{\omega}^2+\varepsilon^2} \Big(2 \operatorname{tr}(W_2 \tilde{W}_4 + \tilde{W}_4 W_2) 1 \\
+ 2 \operatorname{tr}(W_4 - W_2) W_3 + (W_2 - W_4) W_3 + W_3 (W_2 - W_4) - W_2 \tilde{W}_4 - \tilde{W}_4 W_2 \Big)\,,
\end{aligned} \tag{5.1}
$$

where $\underline{\omega} := \omega(k_1) + \omega(k_2) - \omega(k_3) - \omega(k_4)$. By suitably integrating out the convolution δ-function (i.e., by a suitable change of integration variables) we can write its components as a finite sum of integrals of the same form. Explicitly, define for $k_1, k_1', k_2' \in \mathbb{T}^d$,

$$\sigma^{|2} := (1, -1, -1, 1)\,, \qquad k^{|2} := (k_1, k_1' - k_2' - k_1, k_1', -k_2')\,, \tag{5.2}$$

$$\sigma^{|3} := (1, 1, -1, -1)\,, \qquad k^{|3} := (k_1, k_1', k_1 + k_1' - k_2', k_2')\,, \tag{5.3}$$

$$\sigma^{|4} := (1, 1, -1, -1)\,, \qquad k^{|4} := (k_1, k_1', k_2', k_1 + k_1' - k_2')\,, \tag{5.4}$$

and recall the definition of $\tilde{\Omega}(k;\sigma)$ in (2.2). Then, after performing the change of variables as listed in the definition of $\boldsymbol{k}^{|i}$, we find that $\underline{\omega} \to \tilde{\Omega}((\sigma_2^{|i} k_1, k_1', k_2'); \sigma^{|i})$ inside the integrand for all $i = 2, 3, 4$, using the reflection invariance of ω. This shows that any component of $H^{\varepsilon}_{\text{eff}}[W](k_1)$ is a linear combination, with coefficients $\pm\frac{1}{2}$, of a finite number of terms of the form

$$\mathcal{I}^1_\varepsilon[f, g](k_0'; \sigma) := \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1' \mathrm{d}k_2'\, f(k_1') g(k_2') \frac{\tilde{\Omega}}{\tilde{\Omega}^2 + \varepsilon^2} \tag{5.5}$$

where $\tilde{\Omega} = \tilde{\Omega}((k_0', k_1', k_2'); \sigma)$ and in each term $f(k) := W(k)_{ij}$, for some indices i, j, and g is one of the following three choices: $g = 1$, $g(k) = W(k)_{i'j'}$, or $g(k) = W(-k)_{i'j'}$, for some indices i', j'. In addition, for each term there is an $i \in \{2, 3, 4\}$ such that $\sigma = \sigma^{|i}$ and $k_0' = \sigma_2 k_1$. Therefore, for each of the terms, $\|\mathcal{I}^1_\varepsilon[f, g]\|_2$ is equal to the L^2-norm of the term (taken over k_1), and both of the functions f, g belong to $L^2(\mathbb{T}^d)$.

The following results show that the assumptions (DR1)–(DR3) suffice to make also the full collision operator "nicely" continuous in $L^2_{\mathbb{H}}$-norm. For this, we also consider the following approximate collision integrals with possibly discontinuous input,

$$\mathcal{I}^0_\varepsilon[f, g](k_0'; \sigma) := \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1' \mathrm{d}k_2'\, f(k_1') g(k_2') \frac{\varepsilon}{\tilde{\Omega}^2 + \varepsilon^2}\,. \tag{5.6}$$

Proposition 5.1 *Assume that ω satisfies (DR1)–(DR3), and $\sigma \in \{-1, 1\}^4$ is given. Then for any $f, g \in L^\infty(\mathbb{T}^d)$ there are $\mathcal{I}^1_0, \mathcal{I}^0_0 \in L^2(\mathbb{T}^d)$ such that $\mathcal{I}^0_\varepsilon[f, g] \to \mathcal{I}^0_0$, $\mathcal{I}^1_\varepsilon[f, g] \to \mathcal{I}^1_0$ in L^2-norm as $\varepsilon \to 0^+$. For both $j = 0, 1$ and all $\varepsilon \geq 0$, $\mathcal{I}^j_\varepsilon$ is independent of the choice of representatives of f, g, and $\|\mathcal{I}^j_\varepsilon\|_{L^2} \leq C \|f\|_{L^2} \|g\|_{L^\infty}$ with $C := \frac{1}{2} C_{\mathcal{G}}^{1/4}$. There is also a continuous function $u : [0, \infty) \to \mathbb{R}_+$ with $u(0) = 0$ such that $\|\mathcal{I}^j_{\varepsilon'} - \mathcal{I}^j_\varepsilon\|_{L^2} \leq u(\max(\varepsilon', \varepsilon)) \|f\|_{L^2} \|g\|_{L^\infty}$ for every $\varepsilon', \varepsilon > 0$, $f, g \in L^\infty(\mathbb{T}^d)$, $j = 0, 1$.*

In addition, $\mathcal{I}^0_0 \in L^\infty(\mathbb{T}^d)$ with $\|\mathcal{I}^0_0\|_\infty \leq \pi \|\sigma_{\text{coll}}(\cdot, 0; \sigma)\|_\infty \|f\|_\infty \|g\|_\infty$, and using the shorthand $\tilde{\Omega}(k_0', k_1', k_2') := \tilde{\Omega}((k_0', k_1', k_2'); \sigma)$,

$$L^2\!-\!\lim_{\varepsilon \to 0} \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1' \mathrm{d}k_2'\, \frac{\mathbb{1}(|\tilde{\Omega}(\cdot, k_1', k_2')| > \varepsilon)}{\tilde{\Omega}(\cdot, k_1', k_2')} f(k_1') g(k_2') = \mathcal{I}^1_0(\cdot)\,. \tag{5.7}$$

Suppose that $f_\varepsilon, g_\varepsilon \in L^\infty(\mathbb{T}^d)$, $0 < \varepsilon < 1$, are such that $m := \sup_\varepsilon \max(\|f_\varepsilon\|_\infty, \|g_\varepsilon\|_\infty) < \infty$, and $f_\varepsilon \to f$, $g_\varepsilon \to g$ in L^2-norm as $\varepsilon \to 0^+$. Then $f, g \in L^\infty$ and $\|\mathcal{I}^j_\varepsilon[f, g] - \mathcal{I}^j_\varepsilon[f_\varepsilon, g_\varepsilon]\|_{L^2} \leq C_{\mathcal{G}}^{1/4} \frac{m}{2} (\|f - f_\varepsilon\|_{L^2} + \|g - g_\varepsilon\|_{L^2})$ for $\varepsilon > 0$ and $j = 0, 1$. In particular, $\mathcal{I}^0_\varepsilon[f_\varepsilon, g_\varepsilon] \to \mathcal{I}^0_0[f, g]$ and $\mathcal{I}^1_\varepsilon[f_\varepsilon, g_\varepsilon] \to \mathcal{I}^1_0[f, g]$ in L^2-norm as $\varepsilon \to 0^+$.

The key result connecting the limits to the assumption (DR3) is given by the following Lemma.

Lemma 5.2 *Assume that ω is continuous and satisfies (DR3), and $\sigma \in \{-1,1\}^4$, $f,g \in L^\infty(\mathbb{T}^d)$ are given. Suppose $\varphi, \hat\varphi : \mathbb{R} \to \mathbb{C}$ are such that $\hat\varphi \in L^1 \cap L^\infty$ and $\varphi(x) = \int_{-\infty}^{\infty} \mathrm{d}s\, \hat\varphi(s) \mathrm{e}^{\mathrm{i}sx}$ for all $x \in \mathbb{R}$. Define for $k_1 \in \mathbb{T}^d$*

$$\mathcal{I}[f,g](k_1) := \int_{(\mathbb{T}^d)^2} \mathrm{d}k_2 \mathrm{d}k_3\, f(k_2) g(k_3) \varphi(\tilde\Omega(k;\sigma))\,. \tag{5.8}$$

Then $\mathcal{I}[f,g]$ is independent of the choice of the representatives for f,g and $\|\mathcal{I}[f,g]\|_{L^2} \le C_0 \|f\|_{L^2} \|g\|_{L^\infty}$ with $C_0^4 := \int_{\mathbb{R}^4} \mathrm{d}s \prod_{i=1}^4 |\hat\varphi((-1)^{i-1} s_i)|\, |\mathcal{G}(s;\sigma)| \le \|\hat\varphi\|_\infty^4 C_{\mathcal{G}} < \infty$.

Proof: Let us drop σ from the notation, and denote $\tilde\Omega(k) := \tilde\Omega(k;\sigma)$. It follows from the assumptions that φ is continuous, and thus the integral in the definition of $\mathcal{I}[f,g]$ is always convergent, and its value remains invariant if f and g are changed in a set of Lebesgue measure zero. We use Fubini's theorem to integrate over k_3 first, which shows that

$$\|\mathcal{I}[f,g]\|_{L^2}^2 \le \|g\|_\infty^2 \int_{\mathbb{T}^d} \mathrm{d}k_1 \left(\int_{\mathbb{T}^d} \mathrm{d}k_3 \left| \int_{\mathbb{T}^d} \mathrm{d}k_2\, f(k_2) \varphi(\tilde\Omega(k)) \right| \right)^2 . \tag{5.9}$$

By the Cauchy-Schwarz inequality and the normalization $\int_{\mathbb{T}^d} \mathrm{d}k = 1$, the remaining integral is bounded by

$$\int_{(\mathbb{T}^d)^2} \mathrm{d}k_1 \mathrm{d}k_3 \left| \int_{\mathbb{T}^d} \mathrm{d}k_2\, f(k_2) \varphi(\tilde\Omega(k)) \right|^2 . \tag{5.10}$$

By Fubini's theorem, this is equal to

$$\int_{(\mathbb{T}^d)^2} \mathrm{d}k_2 \mathrm{d}k_2'\, f(k_2')^* f(k_2) \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1 \mathrm{d}k_3\, \varphi(\tilde\Omega) \varphi(\tilde\Omega')^*\,, \tag{5.11}$$

where $\tilde\Omega' := \tilde\Omega(k_1, k_2', k_3; \sigma)$. Thus it is bounded by $\|f\|_{L^2}^2$ times the square root of

$$\int_{(\mathbb{T}^d)^2} \mathrm{d}k_2 \mathrm{d}k_2' \left| \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1 \mathrm{d}k_3\, \varphi(\tilde\Omega) \varphi(\tilde\Omega')^* \right|^2 . \tag{5.12}$$

Therefore, we can conclude that the Lemma holds if we can prove that (5.12) can be bounded by C_0^4.

By assumption, we have $\varphi(x) = \int_{-\infty}^{\infty} \mathrm{d}s\, \hat\varphi(s) \mathrm{e}^{\mathrm{i}sx}$ where $\hat\varphi \in L^1$. Therefore, by Fubini's theorem

$$\int_{\mathbb{R}^4} \mathrm{d}s\, \hat\varphi(s_1) \hat\varphi(-s_2)^* \hat\varphi(s_3) \hat\varphi(-s_4)^* \mathcal{G}(s;\sigma) = \int_{(\mathbb{T}^d)^3 \times (\mathbb{T}^d)^3} \mathrm{d}^3k' \mathrm{d}^3k\, \varphi(\Omega_1) \varphi(\Omega_2)^* \varphi(\Omega_3) \varphi(\Omega_4)^*\,, \tag{5.13}$$

where we have used the same notations as in (DR3). However, a second application of Fubini's theorem shows that the right hand side here is equal to (5.12). This implies that (5.12) is bounded by C_0^4 which concludes the proof of the Lemma. □

Proof of Proposition 5.1 Recall the definition of φ^0_ε and $\hat\varphi^0_\varepsilon$ in (3.4), and set also $\varphi^1_\varepsilon(x) := \frac{x}{x^2+\varepsilon^2}$ and $\hat\varphi^1_\varepsilon(x) := \frac{1}{2\mathrm{i}}\operatorname{sign}(x)\mathrm{e}^{-\varepsilon|x|}$, $x\in\mathbb{R}$, $\varepsilon>0$, where $\operatorname{sign}(x)$ denotes the sign of x which we choose to be 0 if $x=0$. By direct integration, we find that both of the pairs $(\varphi^0_\varepsilon,\hat\varphi^0_\varepsilon)$ and $(\varphi^1_\varepsilon,\hat\varphi^1_\varepsilon)$ satisfy the assumptions of Lemma 5.2 for any $\varepsilon>0$. This immediately implies that for any $j=0,1$, $\varepsilon>0$, $\mathcal{I}^j_\varepsilon[f,g]$ is independent of the choice of representatives and $\|\mathcal{I}^j_\varepsilon[f,g]\|_{L^2}\le \frac12 C_{\mathcal{G}}^{1/4}\|f\|_{L^2}\|g\|_\infty<\infty$. Obviously, these properties then also hold for any L^2-limit points.

In fact, then also the pair $(\varphi,\hat\varphi)$ satisfies assumptions of the Lemma, if $j=0,1$, $0<\varepsilon<\varepsilon_0$, and we define $\varphi(x):=\varphi^j_\varepsilon(x)-\varphi^j_{\varepsilon_0}(x)$ and $\hat\varphi(x):=\hat\varphi^j_\varepsilon(x)-\hat\varphi^j_{\varepsilon_0}(x)$ for $x\in\mathbb{R}$. Since then $|\hat\varphi(s)|=\frac12\mathrm{e}^{-\varepsilon|s|}(1-\mathrm{e}^{-(\varepsilon_0-\varepsilon)|s|})\le\frac12(1-\mathrm{e}^{-\varepsilon_0|s|})$, the Lemma implies that

$$\|\mathcal{I}^j_\varepsilon[f,g]-\mathcal{I}^j_{\varepsilon_0}[f,g]\|_{L^2}\le u(\varepsilon_0)\|f\|_{L^2}\|g\|_\infty\,, \tag{5.14}$$

where $u:[0,\infty)\to\mathbb{R}_+$ is defined by

$$u(\varepsilon_0):=\frac12\left(\int_{\mathbb{R}^4}\mathrm{d}s\prod_{i=1}^4\left|1-\mathrm{e}^{-\varepsilon_0|s_i|}\right||\mathcal{G}(s;\sigma)|\right)^{\frac14}. \tag{5.15}$$

Clearly, $u(0)=0$, and dominated convergence theorem can be applied here to prove that u is continuous. This proves the last statement in the first part of the Proposition.

We can conclude that $\|\mathcal{I}^j_\varepsilon[f,g]-\mathcal{I}^j_{\varepsilon'}[f,g]\|_{L^2}\to0$ when $\max(\varepsilon,\varepsilon')\to0^+$. Therefore, if $\varepsilon_n>0$, $n\in\mathbb{N}$, and $\varepsilon_n\to0$, then $\mathcal{I}^j_{\varepsilon_n}[f,g]$ is a Cauchy sequence in L^2. Thus the sequence converges in L^2, and using (5.14) it is straightforward to check that the limit is independent of the choice of the sequence (ε_n). Let the unique limit point be denoted by $\mathcal{I}^j_0[f,g]$. Then the first two statements of the Proposition hold. In addition, taking $\varepsilon\to0$ in (5.14) also shows that for any $\varepsilon_0>0$ and both $j=0,1$

$$\|\mathcal{I}^j_0[f,g]-\mathcal{I}^j_{\varepsilon_0}[f,g]\|_{L^2}\le u(\varepsilon_0)\|f\|_{L^2}\|g\|_\infty\,. \tag{5.16}$$

By the L^2-convergence we can find a sequence $\varepsilon_n>0$, $n\in N$, such that $\varepsilon_n\to0$ and $\mathcal{I}^0_{\varepsilon_n}[f,g](k_0)\to\mathcal{I}^0_0(k_0)$ for almost every k_0. But since then

$$|\mathcal{I}^0_{\varepsilon_n}[f,g](k_0)|\le\|f\|_\infty\|g\|_\infty\int_{(\mathbb{T}^d)^2}\mathrm{d}k_1'\mathrm{d}k_2'\,\frac{\varepsilon_n}{\tilde\Omega^2+\varepsilon_n^2}\overset{n\to\infty}{\longrightarrow}\pi\sigma_{\mathrm{coll}}(k_0,\sigma)\|f\|_\infty\|g\|_\infty\,, \tag{5.17}$$

this implies also $|\mathcal{I}^0_0(k_0)|\le\pi\|\sigma_{\mathrm{coll}}\|_\infty\|f\|_\infty\|g\|_\infty$. As the complement of such k_0 has Lebesgue measure zero, we have proven the claim made about $\mathcal{I}^0_0$ in the Proposition.

To prove (5.7), consider

$$\mathcal{J}^1_\varepsilon(k_0):=\int_{(\mathbb{T}^d)^2}\mathrm{d}k_1\mathrm{d}k_2\,\frac{\mathbb{1}(|\tilde\Omega(k_0,k_1,k_2)|>\varepsilon)}{\tilde\Omega(k_0,k_1,k_2)}f(k_1)g(k_2)\,. \tag{5.18}$$

Obviously, $|\mathcal{J}^1_\varepsilon(k_0)|\le\varepsilon^{-1}\|f\|_\infty\|g\|_\infty$, and thus $\mathcal{J}^1_\varepsilon\in L^2$. We claim that $\|\mathcal{J}^1_\varepsilon-\mathcal{I}^1_{\varepsilon^2}\|_{L^2}\to0$ as $\varepsilon\to0^+$, which implies that $\mathcal{J}^1_\varepsilon\to\mathcal{I}^1_0$ in L^2. For any $x\in\mathbb{R}$,

$$\frac{x}{x^2+\varepsilon^4}-\frac{\mathbb{1}(|x|>\varepsilon)}{x}=-\mathbb{1}(|x|>\varepsilon)\frac{\varepsilon^2}{x}\frac{\varepsilon^2}{x^2+\varepsilon^4}+\mathbb{1}(|x|\le\varepsilon)\frac{x}{x^2+\varepsilon^4}\,. \tag{5.19}$$

The first term is bounded by $\varepsilon^3/(x^2+\varepsilon^4)$, which shows that

$$|\mathcal{J}^1_\varepsilon(k_0) - \mathcal{I}^1_{\varepsilon^2}(k_0) - \mathcal{J}^2_\varepsilon(k_0)| \le \varepsilon \|f\|_\infty \|g\|_\infty \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1 \mathrm{d}k_2 \, \frac{\varepsilon^2}{\tilde\Omega^2+\varepsilon^4} \le \varepsilon \|f\|_\infty \|g\|_\infty C' \,, \tag{5.20}$$

where C' is the constant introduced in the proof of Proposition 3.1, and

$$\mathcal{J}^2_\varepsilon(k_0) := \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1 \mathrm{d}k_2 \, \mathbb{1}(|\tilde\Omega| \le \varepsilon) \frac{\tilde\Omega}{\tilde\Omega^2+\varepsilon^4} f(k_1) g(k_2) \,. \tag{5.21}$$

Therefore, it suffices to prove that $\|\mathcal{J}^2_\varepsilon\|_{L^2} \to 0$ as $\varepsilon \to 0^+$.

For this, define for $\varepsilon, \delta > 0$ the function $\hat\varphi^2_{\varepsilon,\delta} : \mathbb{R} \to \mathbb{C}$ by $\hat\varphi^2_{\varepsilon,\delta}(s) := \frac{1}{\pi \mathrm{i}} \mathrm{e}^{-\delta|s|} h_\varepsilon(s)$ where

$$h_\varepsilon(s) := \int_0^\varepsilon \mathrm{d}\alpha \frac{\alpha}{\alpha^2+\varepsilon^4} \sin(\alpha s) \,. \tag{5.22}$$

Clearly, $h_\varepsilon \in L^\infty(\mathrm{d}s)$ and thus $\hat\varphi^2_{\varepsilon,\delta} \in L^1 \cap L^\infty$ and we can define $\varphi^2_{\varepsilon,\delta}(x) = \int_{-\infty}^\infty \mathrm{d}s \, \hat\varphi^2_{\varepsilon,\delta}(s) \mathrm{e}^{\mathrm{i}sx}$, $x \in \mathbb{R}$. Then Lemma 5.2 can be applied to

$$\mathcal{J}^2_{\varepsilon,\delta}(k_0) := \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1 \mathrm{d}k_2 \, \varphi^2_{\varepsilon,\delta}(\tilde\Omega) f(k_1) g(k_2) \,, \tag{5.23}$$

which shows that $\|\mathcal{J}^2_{\varepsilon,\delta}\|_{L^2} \le c_\varepsilon \pi^{-1} \|f\|_{L^2} \|g\|_{L^\infty}$ with $c_\varepsilon^4 := \int_{\mathbb{R}^4} \mathrm{d}s \prod_{i=1}^4 |h_\varepsilon(|s_i|)| \, |\mathcal{G}(s;\sigma)|$. On the other hand, explicit integration yields for any $x \in \mathbb{R}$

$$\begin{aligned} \varphi^2_{\varepsilon,\delta}(x) &= -\int_{-\infty}^\infty \frac{\mathrm{d}s}{2\pi} \int_{-\varepsilon}^\varepsilon \mathrm{d}\alpha \frac{\alpha}{\alpha^2+\varepsilon^4} \mathrm{e}^{\mathrm{i}s(\alpha+x)-\delta|s|} \\ &= -\int_{-\varepsilon}^\varepsilon \frac{\mathrm{d}\alpha}{2\pi} \frac{\alpha}{\alpha^2+\varepsilon^4} \frac{2\delta}{(\alpha+x)^2+\delta^2} \\ &= \int_{-\infty}^\infty \frac{\mathrm{d}y}{\pi} \frac{1}{1+y^2} \mathbb{1}(|x+\delta y| \le \varepsilon) \frac{x+\delta y}{(x+\delta y)^2+\varepsilon^4} \,, \end{aligned} \tag{5.24}$$

where in the last equality we have changed variables to $y = -(\alpha+x)/\delta$. By dominated convergence, for any $k \in (\mathbb{T}^d)^3$, such that $|\tilde\Omega(k)| \ne \varepsilon$, we have $\lim_{\delta\to 0^+} \varphi^2_{\varepsilon,\delta}(\tilde\Omega) = \mathbb{1}(|\tilde\Omega| \le \varepsilon) \frac{\tilde\Omega}{\tilde\Omega^2+\varepsilon^4}$. By Corollary 3.5, the set of k with $\tilde\Omega(k) = \pm\varepsilon$ has Lebesgue measure zero, and hence we find $\int_{(\mathbb{T}^d)^3} \mathrm{d}^3k |\varphi^2_{\varepsilon,\delta}(\tilde\Omega) - \mathbb{1}(|\tilde\Omega| \le \varepsilon) \frac{\tilde\Omega}{\tilde\Omega^2+\varepsilon^4}|^2 \to 0$ as $\delta \to 0^+$. This implies that $\mathcal{J}^2_{\varepsilon,\delta} \to \mathcal{J}^2_\varepsilon$ in $L^2(\mathbb{T}^2)$, and thus the previous bounds prove that also $\|\mathcal{J}^2_\varepsilon\|_{L^2} \le c_\varepsilon \pi^{-1} \|f\|_{L^2} \|g\|_{L^\infty}$.

Therefore, to conclude the proof of (5.7), we only need to show that $c_\varepsilon \to 0$ when $\varepsilon \to 0^+$. Since $|\sin y| \le |y|$, we have $|h_\varepsilon(s)| \le \varepsilon s$ for any $s \ge 0$, and thus $\lim_{\varepsilon\to 0^+} \prod_{i=1}^4 |h_\varepsilon(|s_i|)| = 0$ for fixed $s \in \mathbb{R}^4$. Now if we can show that $|h_\varepsilon(s)|$ is uniformly bounded for $s \ge 0$ and $\varepsilon > 0$, assumption (DR3) allows to use dominated convergence to prove that $c_\varepsilon \to 0$. If $0 \le s \le 2\varepsilon^{-1}$, we have already found that $|h_\varepsilon(s)| \le 2$. Assume thus $s > 2\varepsilon^{-1}$ and set

$\alpha_0 := \frac{\pi}{2s} \in (0,\varepsilon)$. Then $\left|\int_0^{\alpha_0} \mathrm{d}\alpha \frac{\alpha}{\alpha^2+\varepsilon^4} \sin(\alpha s)\right| \le s\alpha_0 < 2$ and thus it remains to prove that the integral over $[\alpha_0,\varepsilon]$ is bounded. By partial integration and using $\cos(\alpha_0 s) = 0$ we have

$$\int_{\alpha_0}^{\varepsilon} \mathrm{d}\alpha \frac{\alpha}{\alpha^2+\varepsilon^4} \sin(\alpha s) = -\frac{\varepsilon}{\varepsilon^2+\varepsilon^4}\frac{1}{s}\cos(\varepsilon s) + \frac{1}{s}\int_{\alpha_0}^{\varepsilon} \mathrm{d}\alpha \frac{\varepsilon^4-\alpha^2}{(\alpha^2+\varepsilon^4)^2} \cos(\alpha s) \,. \tag{5.25}$$

Here the first term is bounded by $\frac{1}{2}$ and the second by $\frac{1}{s}\int_{\alpha_0}^{\varepsilon} \mathrm{d}\alpha\, \alpha^{-2} \le \frac{1}{s\alpha_0} < 1$. We have shown that always $|h_\varepsilon(s)| \le 4$ which implies $c_\varepsilon \to 0$ and concludes the proof of (5.7).

To prove the second part of the Proposition, assume that $f_\varepsilon, g_\varepsilon \in L^2(\mathbb{T}^d)$, $0 < \varepsilon < 1$, are such that there is $m < \infty$ for which $\|f_\varepsilon\|_\infty, \|g_\varepsilon\|_\infty \le m$, and $f_\varepsilon \to f$, $g_\varepsilon \to g$ in L^2-norm as $\varepsilon \to 0^+$. Since then there is a sequence $\varepsilon_n \to 0^+$ such that $f_{\varepsilon_n} \to f$, $g_{\varepsilon_n} \to g$ pointwise almost everywhere, this implies that also $\|f\|_\infty, \|g\|_\infty \le m$, and thus the previous results can be applied. Telescoping $f(k_2)g(k_3) - f_\varepsilon(k_2)g_\varepsilon(k_3) = (f(k_2) - f_\varepsilon(k_2))g(k_3) + f_\varepsilon(k_2)(g(k_3) - g_\varepsilon(k_3))$ and swapping the integration variables in the second term we can resort to the above bounds and obtain $\|\mathcal{I}^j_\varepsilon[f,g] - \mathcal{I}^j_\varepsilon[f_\varepsilon, g_\varepsilon]\|_{L^2} \le C_{\mathcal{G}}^{1/4}\frac{m}{2}(\|f - f_\varepsilon\|_{L^2} + \|g - g_\varepsilon\|_{L^2})$ for $\varepsilon > 0$. This implies also $\mathcal{I}^j_\varepsilon[f_\varepsilon, g_\varepsilon] \to \mathcal{I}^j_0[f,g]$, as claimed in the Proposition. $\square$

Corollary 5.3 *Assume that ω satisfies (DR1)–(DR3) and $W \in L^2_{\mathit{ferm}}$. Then there is a unique $H_{\mathrm{eff}}[W] \in L^2_{\mathbb{H}}$ such that $H^\varepsilon_{\mathrm{eff}}[W] \to H_{\mathrm{eff}}[W]$ in norm as $\varepsilon \to 0^+$. Moreover, there is a sequence $\varepsilon_n \to 0^+$ such that (2.12) holds for almost every k_1.*

In addition, there is a constant C such that for any $W, W' \in L^2_{\mathit{ferm}}$ and $\varepsilon, \varepsilon' > 0$

$$\|H^\varepsilon_{\mathrm{eff}}[W] - H^{\varepsilon'}_{\mathrm{eff}}[W']\|_2 \le C(\|W' - W\|_2 + u(\max(\varepsilon', \varepsilon))) \,, \tag{5.26}$$

$$\|H_{\mathrm{eff}}[W] - H_{\mathrm{eff}}[W']\|_2 \le C\|W' - W\|_2 \,, \tag{5.27}$$

where u denotes a function satisfying the conclusion of Proposition 5.1.

Proof: As explained in the beginning of this section, any component of $H^\varepsilon_{\mathrm{eff}}[W]$ can be expressed as a finite linear combination of suitably chosen $\mathcal{I}^1_\varepsilon[f,g]$-terms. More precisely, it is straightforward to check that there is a finite index set S such that for any $i,j \in \{1,2\}$ and $n \in S$ we can find a constant $c \in \mathbb{R}$, with $|c| \le 1$, $\ell \in \{2,3,4\}$, $a, a', b, b' \in \{1,2\}$, and $\ell' \in \{0,1,2\}$ such that for all k and W

$$H^\varepsilon_{\mathrm{eff}}[W](k)_{ij} = \sum_{n\in S} c\mathcal{I}^1_\varepsilon[f_{a'b'}, g^{(\ell')}_{ab}](\sigma_2^{|\ell} k; \sigma^{|\ell}) \,, \tag{5.28}$$

where $f_{a'b'}(k) := W(k)_{a'b'}$, $g^{(0)}_{ab}(k) := 1$, $g^{(1)}_{ab}(k) := W(k)_{ab}$, and $g^{(2)}_{ab}(k) := W(-k)_{ab}$. Note that the constants inside the sum can depend on all of n, i, j, even though we have suppressed the dependence from the notation.

If k is such that $0 \le W(k) \le 1$, then $|W(k)_{ij}| \le 1$ for all i,j. Thus it follows that in (5.28) always $f, g \in L^\infty(\mathbb{T}^d)$ with $\|f\|_\infty, \|g\|_\infty \le 1$. Thus we can apply Proposition 5.1 and conclude that $H^\varepsilon_{\mathrm{eff}}[W]_{ij} \to H_{\mathrm{eff}}[W]_{ij}$ in $L^2(\mathbb{T}^d)$ as $\varepsilon \to 0^+$, where

$$H_{\mathrm{eff}}[W](k)_{ij} := \sum_{n\in S} c\mathcal{I}^1_0[f_{a'b'}, g^{(\ell')}_{ab}](\sigma_2^{|\ell} k; \sigma^{|\ell}) \,. \tag{5.29}$$

Let $H_{\rm eff}[W](k)$ denote the matrix collecting the above limit functions. Then we can find a sequence $\varepsilon_n \to 0$ such that $H^{\varepsilon_n}_{\rm eff}[W](k)_{ij} \to H_{\rm eff}[W](k)_{ij}$ for all i,j and almost every k. As $H^{\varepsilon}_{\rm eff}[W]_{ij}(k)^* = H^{\varepsilon}_{\rm eff}[W]_{ji}(k)$ for every k, this implies that $H_{\rm eff}[W](k)$ is then Hermitian at almost every k. In addition, $\|H^{\varepsilon}_{\rm eff}[W] - H_{\rm eff}[W]\|_2^2 = \sum_{i,j} \|H^{\varepsilon}_{\rm eff}[W]_{ij} - H_{\rm eff}[W]_{ij}\|^2_{L^2}$, and therefore $H_{\rm eff}[W] \in L^2_{\mathbb{H}}$ and $H^{\varepsilon}_{\rm eff}[W] \to H_{\rm eff}[W]$ as $\varepsilon \to 0$. Since the index set S is finite, the representation of the limit as a standard principal value integral, (2.12), follows then from an application of (5.7) to (5.29).

Suppose then that $W, W' \in L^2_{\rm ferm}$ and $\varepsilon, \varepsilon' > 0$. Then $\|H^{\varepsilon}_{\rm eff}[W] - H^{\varepsilon'}_{\rm eff}[W']\|_2 \le \|H^{\varepsilon}_{\rm eff}[W] - H^{\varepsilon}_{\rm eff}[W']\|_2 + \|H^{\varepsilon}_{\rm eff}[W'] - H^{\varepsilon'}_{\rm eff}[W']\|_2$. Let u be a function as in Proposition 5.1. We can conclude that $\|H^{\varepsilon}_{\rm eff}[W']_{ij} - H^{\varepsilon'}_{\rm eff}[W']_{ij}\|_2 \le |S| u(\max(\varepsilon,\varepsilon'))\|W'\|_2$ for any i,j. Here $\|W'\|_2^2 \le 2$, since $W' \in L^2_{\rm ferm}$, and thus $\|H^{\varepsilon}_{\rm eff}[W'] - H^{\varepsilon'}_{\rm eff}[W']\|_2 \le 4|S|u(\max(\varepsilon,\varepsilon'))$.

To study $\|H^{\varepsilon}_{\rm eff}[W] - H^{\varepsilon}_{\rm eff}[W']\|_2$ write $\Delta := W - W'$ and note that then $f_{ab}[W] = f_{ab}[W'] + f_{ab}[\Delta]$, $g^{(\ell')}_{ab}[W] = g^{(\ell')}_{ab}[W'] + g^{(\ell')}_{ab}[\Delta]$, for $\ell' = 1,2$, and $g^{(0)}_{ab}[W] = g^{(0)}_{ab}[W']$. Since $\mathcal{I}^1_\varepsilon[f,g]$ is obviously linear in both f and g, this shows that

$$
\begin{aligned}
H^{\varepsilon}_{\rm eff}[W](k)_{ij} - H^{\varepsilon}_{\rm eff}[W'](k)_{ij} &= \sum_{n\in S} c\mathcal{I}^1_\varepsilon[\Delta_{a'b'}, g^{(\ell')}_{ab}[W]](\sigma_2^{|\ell} k; \sigma^{|\ell}) \\
&+ \sum_{n\in S} c\mathbb{1}(\ell' \ne 0)\mathcal{I}^1_\varepsilon[W'_{a'b'}, g^{(\ell')}_{ab}[\Delta]](\sigma_2^{|\ell} k; \sigma^{|\ell}) \,. \qquad (5.30)
\end{aligned}
$$

Consider then one term in the second sum here. If $\ell' = 0$, then the term is zero. If $\ell' = 1$, we swap the integration variables, and if $\ell' = 2$, we change the integration variables to $k''_1 = -k'_2$ and $k''_2 = -k'_1$. This shows that each term in the second sum is either zero or of the form $c\mathcal{I}^1_\varepsilon[\Delta_{ab}, g^{(\ell')}_{a'b'}[W']](\pm k; \sigma')$ for some choice of the sign and σ'. Applying Proposition 5.1, this shows that $\|H^{\varepsilon}_{\rm eff}[W](k)_{ij} - H^{\varepsilon}_{\rm eff}[W'](k)_{ij}\|_{L_2} \le |S| C_{\mathcal{G}}^{1/4}\|\Delta\|_2$. Therefore, $\|H^{\varepsilon}_{\rm eff}[W] - H^{\varepsilon}_{\rm eff}[W']\|_2 \le 2|S|^{1/2} C_{\mathcal{G}}^{1/4}\|W - W'\|_2$.

Thus by choosing C as larger of the constants in these two bounds, we can conclude that (5.26) holds. Then taking $\varepsilon, \varepsilon' \to 0$ in (5.26) proves also (5.27) and concludes the proof of the corollary. □

Finally, let us show that adding the assumption (DR3) simplifies the definition of the dissipative term.

Corollary 5.4 *Assume that ω satisfies (DR1)–(DR3). Then for any $w_i \in L^\infty(\mathbb{T}^d)$, $i = 1,2,3$, we have*

$$
\mathcal{C}_0[w_1,w_2,w_3](k_1) = \lim_{\varepsilon\to 0^+} \int_{(\mathbb{T}^d)^3} \frac{{\rm d}k_2 {\rm d}k_3 {\rm d}k_4}{\pi} \delta(\underline{k}) \frac{\varepsilon}{\varepsilon^2 + \underline{\omega}^2} w_1(k_2) w_2(k_3) w_3(k_4)\,, \qquad (5.31)
$$

where the limit converges in $L^2(\mathbb{T}^d)$-norm.

Therefore, then for any $W \in L^2_{ferm}$ and $i,j \in \{1,2\}$ the L^2-limit in (2.10) holds.

Proof: For the given w_i, set $m := \max \|w_i\|_\infty$, and choose a sequence $v_{i,n}$ as explained at the end of the proof of Lemma 3.3. In particular, then $v_{i,n} \to w_i$ and $\mathcal{C}_{0,n} \to \mathcal{C}_0$ in L^2-norm as $n \to \infty$, where $\mathcal{C}_{0,n} := \mathcal{C}_0[v_{1,n}, v_{2,n}, v_{3,n}]$ and $\mathcal{C}_0 := \mathcal{C}_0[w_1, w_2, w_3]$. For any $\varepsilon > 0$ and k_1 denote the value of the integral on the right hand side of (5.31) by $\mathcal{C}_\varepsilon(k_1)$, and define $\mathcal{C}_{\varepsilon,n} := \mathcal{C}_\varepsilon[v_{1,n}, v_{2,n}, v_{3,n}]$ analogously. Then the function $\mathcal{C}_\varepsilon \in L^\infty$, and we need to prove that $\|\mathcal{C}_0 - \mathcal{C}_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0^+$.

For any n, ε we can estimate $\|\mathcal{C}_0 - \mathcal{C}_\varepsilon\|_2 \le \|\mathcal{C}_0 - \mathcal{C}_{0,n}\|_2 + \|\mathcal{C}_{0,n} - \mathcal{C}_{\varepsilon,n}\|_2 + \|\mathcal{C}_{\varepsilon,n} - \mathcal{C}_\varepsilon\|_2$. By Lemma 3.3, here $\|\mathcal{C}_0 - \mathcal{C}_{0,n}\|_2 \le Cm^2 \sum_{i=1}^3 \|w_i - v_{i,n}\|_2$. Using a similar telescoping estimate and Proposition 5.1, we can also find a constant C' such that $\|\mathcal{C}_{\varepsilon,n} - \mathcal{C}_\varepsilon\|_2 \le C'm^2 \sum_{i=1}^3 \|w_i - v_{i,n}\|_2$. Hence, for any $\varepsilon_0 > 0$ we can find n such that $\|\mathcal{C}_0 - \mathcal{C}_\varepsilon\|_2 \le \frac{\varepsilon_0}{3} + \|\mathcal{C}_{0,n} - \mathcal{C}_{\varepsilon,n}\|_2$. However, by Proposition 3.1, we can use dominated converge to conclude that $\|\mathcal{C}_{0,n} - \mathcal{C}_{\varepsilon,n}\|_2 \to 0$ as $\varepsilon \to 0^+$. In particular, there is $\varepsilon' > 0$ such that for all $0 < \varepsilon < \varepsilon'$ it is less than $\frac{\varepsilon_0}{3}$. This proves that $\|\mathcal{C}_0 - \mathcal{C}_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0^+$.

Suppose then that $W \in L^2_{\text{ferm}}$ and $i, j \in \{1, 2\}$ and consider the definition of $\mathcal{C}_{\text{diss}}[W]_{ij}$ given in (3.23). Each of the $\mathcal{C}_0$-terms in the finite sum can be approximated in L^2-norm by the corresponding $\mathcal{C}_\varepsilon$-integral, for some fixed $\varepsilon > 0$. The resulting $\mathcal{C}_\varepsilon$-terms then sum to the right hand side of (2.10), and hence the limit holds in L^2-norm. □

Proof of Theorems 2.2 and 2.3 Proposition 3.1 and Corollary 3.4 imply the statements in the first paragraph of Theorem 2.2, as well as the first statement of the second paragraph. Corollary 5.4 proves the remaining statements in the second paragraph and completes the proof of Theorem 2.2.

Theorem 2.3 is a direct consequence of Corollary 5.3. □

6 Proof of Theorem 2.4

Assume now that all of the conditions (DR1)–(DR3) hold, and define $\mathcal{C}_{\text{diss}}$ and H_{eff} as in the already proven Theorems 2.2 and 2.3. For any representative of $W \in L^2_{\text{ferm}}$ and $0 < \delta < 1$ let us define the corresponding "locally averaged representative" $A_\delta[W]$ by setting for $k \in \mathbb{T}^d$,

$$A_\delta[W](k) := \frac{1}{Z_\delta} \int_{\mathbb{T}^d} \mathrm{d}k'\, \mathbb{1}(|k'| \le \delta) W(k + k')\,, \tag{6.1}$$

where $Z_\delta := \int_{\mathbb{T}^d} \mathrm{d}k\, \mathbb{1}(|k| \le \delta)$ is the appropriate normalization factor. The formula is understood as a vector-valued integral over the compact set $\{k \in \mathbb{T}^d \,|\, |k| \le \delta\}$ in the Banach space $L^2_{\mathbb{H}}$. Since $L^2_{\text{ferm}} \subset L^\infty(\mathbb{T}^d, \mathbb{C}^{2\times 2})$, we can use dominated converge to conclude that $A_\delta[W]$ is continuous on $\mathbb{T}^d$. Also the Fermi property is clearly preserved and thus A_δ is a linear map from L^2_{ferm} to X_{ferm}.

We can also identify any component function $W_{ij}(k)$ with a function in $L^1(\mathbb{R}^d)$ by extending it periodically to the neighboring "cells" and then setting the extension to be zero in all other cells. If k is a Lebesgue point of this extension, then $A_\delta[W]_{ij}(k) \to W_{ij}(k)$ as $\delta \to 0$, cf. [19, Theorem 7.10]. Since almost every point is then a Lebesgue point, we find that $\lim_{\delta\to 0} \|A_\delta[W](k) - W(k)\|^2 = 0$ apart possibly from a set of measure zero collecting the non-Lebesgue points of the three independent component functions of W. Since $\|A_\delta[W]\|_\infty \le \|W\|_\infty < \infty$, dominated convergence implies that $A_\delta[W] \to W$ in $L^2_{\mathbb{H}}$.

Let us then consider some allowed initial data $W_0 \in L^2_{\rm ferm}$ and some allowed regulators $0 < \varepsilon, \delta < 1$. By Theorem 4.1, for all $t \ge 0$, we then have a unique solution $w_t(\cdot; \varepsilon, \delta, W_0) \in X_{\rm ferm}$ to the regularized problem with initial data $w_0 = A_\delta[W_0]$. We can also find for $k \in \mathbb{T}^d$ and $0 \le s \le t$ unitary matrices $u_{t,s}(k; \varepsilon, \delta, W_0)$ such that $u_{t,t}(k) = 1$ and $s \mapsto u_{t,s}(k)$ belongs to $C^{(1)}([0,t], \mathbb{C}^{2\times 2})$ with $\partial_s u_{t,s}(k) = {\rm i} u_{t,s}(k) H^\varepsilon_{\rm eff}[w_s](k)$, and we have for all $t \ge 0$, and k,

$$w_t(k) = u_{t,0}(k) w_0(k) u_{t,0}(k)^* + \int_0^t {\rm d}s\, u_{t,s}(k) \mathcal{C}_{\rm diss}[w_s](k) u_{t,s}(k)^* \,. \tag{6.2}$$

Let us then consider two allowed initial data $W_0', W_0 \in L^2_{\rm ferm}$ and some allowed regulators $0 < \varepsilon', \varepsilon, \delta', \delta < 1$. Define w and u as above using ε, δ and W_0, and set also $w_t'(k) := w_t(k; \varepsilon', \delta', W_0')$ and $u_{t,s}'(k) := u_{t,s}(k; \varepsilon', \delta', W_0')$. Using (6.2) for both solutions and telescoping shows that then

$$\begin{aligned} w_t(k) - w_t'(k) &= u_{t,0}(k) \left(\delta w_0(k) + \delta u_{t,0}(k) w_0'(k)\right) u_{t,0}(k)^* - u_{t,0}'(k) w_0'(k) \delta u_{t,0}(k) u_{t,0}'(k)^* \\ &\quad + \int_0^t {\rm d}s\, u_{t,s}(k) \left(\mathcal{C}_{\rm diss}[w_s](k) - \mathcal{C}_{\rm diss}[w_s'](k)\right) u_{t,s}(k)^* \\ &\quad + \int_0^t {\rm d}s\, u_{t,s}(k) \delta u_{t,s}(k) \mathcal{C}_{\rm diss}[w_s'](k) u_{t,s}(k)^* - \int_0^t {\rm d}s\, u_{t,s}'(k) \mathcal{C}_{\rm diss}[w_s'](k) \delta u_{t,s}(k) u_{t,s}'(k)^* \,, \end{aligned} \tag{6.3}$$

where $\delta u_{t,s}(k) := 1 - u_{t,s}(k)^* u_{t,s}'(k)$ and $\delta w_0(k) := A_\delta[W_0] - A_{\delta'}[W_0']$. Clearly, $\|UMU^*\| = \|M\|$ if U is a unitary matrix. Thus we obtain from (6.3) a bound

$$\begin{aligned} \|w_t(k) - w_t'(k)\| &\le \|\delta w_0(k)\| + 2\|\delta u_{t,0}(k)\| \|w_0'(k)\| \\ &\quad + \int_0^t {\rm d}s\, \|\mathcal{C}_{\rm diss}[w_s](k) - \mathcal{C}_{\rm diss}[w_s'](k)\| + 2\int_0^t {\rm d}s\, \|\delta u_{t,s}(k)\| \|\mathcal{C}_{\rm diss}[w_s'](k)\| \,. \end{aligned} \tag{6.4}$$

Here $\|w_0'(k)\| = \|A_{\delta'}[W_0'](k)\| \le \operatorname{ess\,sup}_{k'} \|W_0'(k')\| \le 2$, since $0 \le W_0' \le 1$ almost everywhere. Therefore, we find a bound

$$\begin{aligned} \|w_t - w_t'\|_2 &\le \|\delta w_0\|_2 + 4\|\delta u_{t,0}\|_2 + \int_0^t {\rm d}s\, \|\mathcal{C}_{\rm diss}[w_s] - \mathcal{C}_{\rm diss}[w_s']\|_2 \\ &\quad + 2\int_0^t {\rm d}s\, \|\delta u_{t,s}\|_2 \operatorname{ess\,sup}_k \|\mathcal{C}_{\rm diss}[w_s'](k)\| \,. \end{aligned} \tag{6.5}$$

Here we can apply Corollary 3.4, which implies that there is a pure constant C such that

$$\|w_t - w_t'\|_2 \le \|\delta w_0\|_2 + 4\|\delta u_{t,0}\|_2 + C\int_0^t {\rm d}s\, \|w_s - w_s'\|_2 + 2C\int_0^t {\rm d}s\, \|\delta u_{t,s}\|_2 \,. \tag{6.6}$$

For simplicity, denote $h_s(k) := H^\varepsilon_{\rm eff}[w_s](k)$ and $h_s'(k) := H^{\varepsilon'}_{\rm eff}[w_s'](k)$. Now $\delta u_{t,t}(k) = 0$ and $\|\delta u_{t,s}(k)\|^2 = 2\,{\rm tr}\, 1 - {\rm tr}(u_{t,s}(k) u_{t,s}'(k)^* + u_{t,s}'(k) u_{t,s}(k)^*)$. Thus $\partial_s \|\delta u_{t,s}(k)\|^2 =$

$-\mathrm{i}\,\mathrm{tr}\left[(h_s(k)-h'_s(k))u'_{t,s}(k)^*u_{t,s}(k)\right]+\mathrm{i}\,\mathrm{tr}\left[(h_s(k)-h'_s(k))u_{t,s}(k)^*u'_{t,s}(k)\right]$. Applying Cauchy-Schwarz inequality to the Hilbert-Schmidt scalar product here, we find that $|\partial_s\|\delta u_{t,s}(k)\|^2|\le \|u'_{t,s}(k)^*u_{t,s}(k)-u_{t,s}(k)^*u'_{t,s}(k)\|\|h_s(k)-h'_s(k)\|\le 2\|\delta u_{t,s}(k)\|\|h_s(k)-h'_s(k)\|$. Therefore, for any $0\le r\le t$ we have $\|\delta u_{t,r}(k)\|^2=-\int_r^t \mathrm{d}s\,\partial_s\|\delta u_{t,s}(k)\|^2\le 2\int_r^t\mathrm{d}s\,\|\delta u_{t,s}(k)\|\|h_s(k)-h'_s(k)\|$. Set $m_t(k):=\sup_{0\le s\le t}\|\delta u_{t,s}(k)\|$, where obviously $m_t(k)\le 4<\infty$. It follows that $\|\delta u_{t,r}(k)\|^2\le 2m_t(k)\int_0^t\mathrm{d}s\,\|h_s(k)-h'_s(k)\|$, and hence $m_t(k)^2\le 2m_t(k)\int_0^t\mathrm{d}s\,\|h_s(k)-h'_s(k)\|$. Thus we can conclude that for all $0\le r\le t$ we have

$$\|\delta u_{t,r}(k)\|\le 2\int_0^t \mathrm{d}s\,\|H^{\varepsilon}_{\mathrm{eff}}[w_s](k)-H^{\varepsilon'}_{\mathrm{eff}}[w'_s](k)\|\,. \tag{6.7}$$

Therefore, by using the constant C and the function $u(x)$ as in Corollary 5.3, we can conclude that $\|\delta u_{t,r}\|_2\le 2C\int_0^t\mathrm{d}s\,(\|w_s-w'_s\|_2+u(\max(\varepsilon,\varepsilon')))$. Applied in (6.6), this shows that there is a pure constant C' such that

$$\|w_t-w'_t\|_2\le\|\delta w_0\|_2+C't(1+t)u(\max(\varepsilon,\varepsilon'))+C'(1+t)\int_0^t\mathrm{d}s\,\|w_s-w'_s\|_2\,. \tag{6.8}$$

Since the map $t\mapsto\|w_t-w'_t\|_2$ is continuous, Grönwall's lemma can be applied here, and we can conclude that, if $t_0>0$, then for all $0\le t\le t_0$ we have

$$\|w_t-w'_t\|_2\le(\|A_\delta[W_0]-A_{\delta'}[W'_0]\|_2+C't(1+t)u(\max(\varepsilon,\varepsilon')))\,\mathrm{e}^{C'(1+t_0)t}\,. \tag{6.9}$$

We can thus apply this also in the special case $W'_0=W_0$ and consider the sequence of solutions defined for $\varepsilon=1/n$, $\delta=1/n$: set $W_{t,n}(k):=w_t(k;1/n,1/n,W_0)$, $n\in\mathbb{N}$. By (6.9), if $0\le t\le t_0$, then

$$\|W_{t,n}-W_{t,n'}\|_2\le\left(\|A_{1/n}[W_0]-A_{1/n'}[W_0]\|_2+C't_0(1+t_0)u(1/\min(n,n')\right)\mathrm{e}^{C'(1+t_0)t_0}\,. \tag{6.10}$$

Thus $W_{t,n}$ forms a Cauchy sequence in $L^2_{\mathbb{H}}$ and there exists $W_t:=\lim_{n\to\infty}W_{t,n}$ for all $t\ge 0$ (by the previous discussion, the limit indeed coincides with W_0 if $t=0$). Since $W_{t,n}\in L^2_{\mathrm{ferm}}$ for all n and L^2_{ferm} is a closed subset, this implies that $W_t\in L^2_{\mathrm{ferm}}$. By Corollary 5.3, then also

$$H_{\mathrm{eff}}[W_t]=\lim_{n\to\infty}H^{\frac{1}{n}}_{\mathrm{eff}}[W_t]=\lim_{n\to\infty}H^{\frac{1}{n}}_{\mathrm{eff}}[W_{t,n}]\,, \tag{6.11}$$

where the limits are taken in L^2-norm. The conservation laws (2.14) and (2.15) can be identified as scalar products in $L^2_{\mathbb{H}}$, the first with the function $k\mapsto\omega(k)1$, and the second with the constant matrix $M_{i'j'}:=\delta_{ij'}\delta_{ji'}$. Since the equalities hold for all $W_{t,n}$ and these converge in $L^2_{\mathbb{H}}$ to W_t, this implies (2.14) and (2.15).

Since $W_{t,n}$ is a solution to the regularized problem, it satisfies a pointwise identity

$$W_{t,n}(k)=W_{0,n}(k)+\int_0^t\mathrm{d}s\,\left(\mathcal{C}_{\mathrm{diss}}[W_{s,n}](k)-\mathrm{i}[H^{1/n}_{\mathrm{eff}}[W_{s,n}](k),W_{s,n}(k)]\right)\,. \tag{6.12}$$

By taking a scalar product with an arbitrary element in $L^2_{\mathbb{H}}$ and using Corollaries 3.4 and 5.3, we thus find that as vector valued integrals in $L^2_{\mathbb{H}}$

$$W_t = W_0 + \int_0^t \mathrm{d}s \, \left(\mathcal{C}_{\mathrm{diss}}[W_s] - \mathrm{i}[H_{\mathrm{eff}}[W_s], W_s] \right) . \tag{6.13}$$

This implies first that $t \mapsto W_t$ is in $C([0,\infty), L^2_{\mathrm{ferm}})$ and the Fréchet derivative satisfies $\partial_t W_t = \mathcal{C}[W_t]$ for $t > 0$. Thus $W \in C^{(1)}([0,\infty), L^2_{\mathrm{ferm}})$.

Finally, suppose W'_t is an arbitrary solution in $C^{(1)}([0,\infty), L^2_{\mathrm{ferm}})$ with initial data W'_0, and denote $\Delta_t := W_t - W'_t$. Then $\Delta_t \in C^{(1)}([0,\infty), L^2_{\mathbb{H}})$ and

$$\partial_t \Delta_t = \mathcal{C}_{\mathrm{diss}}[W_t] - \mathcal{C}_{\mathrm{diss}}[W'_t] - \mathrm{i}[H_{\mathrm{eff}}[W_t] - H_{\mathrm{eff}}[W'_t], W'_t] - \mathrm{i}[H_{\mathrm{eff}}[W_t], \Delta_t] \, . \tag{6.14}$$

For this computation, and those that follow, it can be helpful to note that $L^2_{\mathbb{H}}$ is a real Hilbert space whose scalar product can be written as $(w', w) = \int \mathrm{d}k \, \mathrm{tr}(w'(k) w(k))$. In particular, it can be applied to conclude that $t \mapsto \|\Delta_t\|_2^2$ is continuously differentiable with $\partial_t \|\Delta_t\|_2^2 = 2(\Delta_t, \partial_t \Delta_t)$. Since by periodicity of trace we have $\mathrm{tr}(\Delta_t(k)[H_{\mathrm{eff}}[W_t](k), \Delta_t(k)]) = 0$ for all k , this shows that

$$\partial_t \|\Delta_t\|_2^2 = 2 \left(\Delta_t, \mathcal{C}_{\mathrm{diss}}[W_t] - \mathcal{C}_{\mathrm{diss}}[W'_t] - \mathrm{i}[H_{\mathrm{eff}}[W_t] - H_{\mathrm{eff}}[W'_t], W'_t] \right) . \tag{6.15}$$

Applying Corollaries 3.4 and 5.3, the property $\|W'_t\| \le 4$, and Cauchy-Schwarz inequality, we can conclude that there is a constant C such that $|\partial_t \|\Delta_t\|_2^2| \le C \|\Delta_t\|_2^2$. Therefore, Grönwall's lemma implies that for all $t \ge 0$

$$\|W_t - W'_t\|_2^2 \le \|W_0 - W'_0\|_2^2 \mathrm{e}^{Ct} \, . \tag{6.16}$$

This result implies both the stated stability and uniqueness of solutions in L^2_{ferm}, and thus concludes the proof of the Theorem. □

A Nearest neighbor dispersion relation at $d \ge 3$

We prove here that the nearest neighbor dispersion relation of the square lattice of dimension $d \ge 3$ satisfies all of the assumptions of the main theorem.

Proposition A.1 *If $d \ge 3$, all the properties listed in Assumption 2.1 are satisfied by the nearest neighbor dispersion relation,*

$$\omega(k) := c - \sum_{\nu=1}^{d} \cos p^\nu , \quad \textit{with } p = 2\pi k, \tag{A.1}$$

where $c \in \mathbb{R}$ is arbitrary.

Proof: Fix some $d \geq 3$ and define ω by (A.1). It is clear that ω is then continuous and satisfies $\omega(-k) = \omega(k)$. In the Appendix to [12] it is already proven that then there is a constant C such that the free propagator $p_t(x)$ satisfies $\|p_t\|^3 \leq C(1+|t|)^{-\frac{3d}{7}}$, and hence it belongs to $L^1(\mathrm{d}t)$. Therefore, (DR1) and (DR2) hold and we only need to prove that also (DR3) is satisfied (note that the present conditions are different from those defined in [12]).

Fix $\sigma \in \{-1,1\}^4$ and recall the definition of the functions $\tilde{\Omega}$ and Ω_i given in the statement of (DR3). We need to inspect

$$\mathcal{G}(s) = \int_{(\mathbb{T}^d)^3 \times (\mathbb{T}^d)^3} \mathrm{d}^3k' \mathrm{d}^3k \, \mathrm{e}^{\mathrm{i} \sum_{i=1}^4 s_i \Omega_i(k,k')} , \tag{A.2}$$

defined for $s \in \mathbb{R}^4$. It suffices to prove that $\int_{\mathbb{R}^4} \mathrm{d}s |\mathcal{G}(s)| < \infty$, as we can then choose as $C_{\mathcal{G}}$ the maximum of all such bounds obtained from the 16 possible choices of σ.

For the nearest neighbor dispersion relation, the constant term produces only a global phase factor, and the integrals corresponding to the d different "directions" factorize. This shows that $|\mathcal{G}(s)| = |F(s)|^d \leq |F(s)|^3$, where

$$F(s) := \int_{\mathbb{T}^3 \times \mathbb{T}^3} \mathrm{d}^3k' \mathrm{d}^3k \, \mathrm{e}^{-\mathrm{i}g(s,k,k')} , \tag{A.3}$$

$$g(s,k,k') := \sum_{i,j=1}^4 s_i \sigma_j \cos(P_{ij}(k,k')) = \mathrm{Re}\left(\sum_{i,j=1}^4 s_i \sigma_j \mathrm{e}^{\mathrm{i}P_{ij}(k,k')} \right) . \tag{A.4}$$

Each P_{ij} is a linear function of (k,k'), which are easiest to define by using the following matrix representation

$$P(k,k') := 2\pi \begin{pmatrix} k_1 & k_2 & k_3 & k_1+k_2-k_3 \\ k_1 & k_2' & k_3 & k_1+k_2'-k_3 \\ k_1' & k_2 & k_3' & k_1'+k_2-k_3' \\ k_1' & k_2' & k_3' & k_1'+k_2'-k_3' \end{pmatrix} . \tag{A.5}$$

In order to better decouple the interdependence, let us change the integration variables from (k,k') to (q,α) where $q_1 = 2\pi k_1$, $q_2 = 2\pi k_1'$, $q_3 = 2\pi k_2$, $\alpha_1 = 2\pi(k_3 - k_2)$, $\alpha_2 = 2\pi(k_3' - k_2')$ and $\alpha_3 = 2\pi(k_2 - k_2')$. Then

$$P(k,k') = \begin{pmatrix} q_1 & q_3 & q_3+\alpha_1 & q_1-\alpha_1 \\ q_1 & q_3-\alpha_3 & q_3+\alpha_1 & q_1-\alpha_1-\alpha_3 \\ q_2 & q_3 & q_3+\alpha_2-\alpha_3 & q_2+\alpha_3-\alpha_2 \\ q_2 & q_3-\alpha_3 & q_3+\alpha_2-\alpha_3 & q_2-\alpha_2 \end{pmatrix} , \tag{A.6}$$

and, therefore,

$$\begin{aligned} g(s,k,k') = \mathrm{Re}\Big(& \mathrm{e}^{\mathrm{i}q_1} \left[s_1\sigma_1 + s_2\sigma_1 + s_1\sigma_4 \mathrm{e}^{-\mathrm{i}\alpha_1} + s_2\sigma_4 \mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_3)} \right] \\ & + \mathrm{e}^{\mathrm{i}q_2} \left[s_3\sigma_1 + s_4\sigma_1 + s_3\sigma_4 \mathrm{e}^{\mathrm{i}(\alpha_3-\alpha_2)} + s_4\sigma_4 \mathrm{e}^{-\mathrm{i}\alpha_2} \right] \\ & + \mathrm{e}^{\mathrm{i}q_3} \big[s_1\sigma_2 + s_2\sigma_2 \mathrm{e}^{-\mathrm{i}\alpha_3} + s_3\sigma_2 + s_4\sigma_2 \mathrm{e}^{-\mathrm{i}\alpha_3} \\ & \quad + s_1\sigma_3 \mathrm{e}^{\mathrm{i}\alpha_1} + s_2\sigma_3 \mathrm{e}^{\mathrm{i}\alpha_1} + s_3\sigma_3 \mathrm{e}^{\mathrm{i}(\alpha_2-\alpha_3)} + s_4\sigma_3 \mathrm{e}^{\mathrm{i}(\alpha_2-\alpha_3)} \big] \Big) . \end{aligned} \tag{A.7}$$

This shows that for any fixed $s \in \mathbb{R}^4$ and $\alpha \in [-\pi, \pi]^3$ there is $\varphi(s, \alpha) \in \mathbb{R}^3$ such that

$$g(s, k, k') = \sum_{i=1}^{3} R_i(s, \alpha) \cos(q_i + \varphi_i(s, \alpha)), \tag{A.8}$$

where

$$R_1(s, \alpha) := \left| s_1\sigma_1 + s_2\sigma_1 + s_1\sigma_4 \mathrm{e}^{-\mathrm{i}\alpha_1} + s_2\sigma_4 \mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_3)} \right|, \tag{A.9}$$

$$R_2(s, \alpha) := \left| s_3\sigma_1 + s_4\sigma_1 + s_3\sigma_4 \mathrm{e}^{\mathrm{i}(\alpha_3-\alpha_2)} + s_4\sigma_4 \mathrm{e}^{-\mathrm{i}\alpha_2} \right|, \tag{A.10}$$

$$\begin{aligned} R_3(s, \alpha) := \big| & s_1\sigma_2 + s_2\sigma_2 \mathrm{e}^{-\mathrm{i}\alpha_3} + s_3\sigma_2 + s_4\sigma_2 \mathrm{e}^{-\mathrm{i}\alpha_3} \\ & + s_1\sigma_3 \mathrm{e}^{\mathrm{i}\alpha_1} + s_2\sigma_3 \mathrm{e}^{\mathrm{i}\alpha_1} + s_3\sigma_3 \mathrm{e}^{\mathrm{i}(\alpha_2-\alpha_3)} + s_4\sigma_3 \mathrm{e}^{\mathrm{i}(\alpha_2-\alpha_3)} \big| . \end{aligned} \tag{A.11}$$

Therefore, we can first integrate over q, and this yields

$$F(s) = \int_{[-\pi,\pi]^3} \frac{\mathrm{d}\alpha}{(2\pi)^3} \prod_{i=1}^{3} f(R_i(s, \alpha)), \tag{A.12}$$

where $f(r) := \int_{-\pi}^{\pi} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{-\mathrm{i}r\cos p}$. As shown in [12], by a direct saddle point argument, here we can find a pure constant C_1 such that $|f(r)| \le C_1(1 + |r|)^{-\frac{1}{2}}$.

Collecting all the above estimates together, we can conclude that

$$\int_{\mathbb{R}^4} \mathrm{d}s |\mathcal{G}(s)| \le \left(\frac{C_1}{2\pi}\right)^9 \int_{\mathbb{R}^4} \mathrm{d}s \left(\int_{[-\pi,\pi]^3} \mathrm{d}\alpha \prod_{i=1}^{3} (1 + R_i(s, \alpha))^{-\frac{1}{2}} \right)^3 . \tag{A.13}$$

By Fubini's theorem, even if infinite, the integral here is equal to

$$\int_{([-\pi,\pi]^3)^3} \mathrm{d}\alpha^{(1)} \mathrm{d}\alpha^{(2)} \mathrm{d}\alpha^{(3)} \int_{\mathbb{R}^4} \mathrm{d}s \prod_{j,\ell=1}^{3} f_{\ell,j}(s)^{-\frac{1}{4}}, \tag{A.14}$$

where for all $i, j \in \{1, 2, 3\}$ we define

$$m_{i,j}(s) := 1 + R_i(s, \alpha^{(j)}), \tag{A.15}$$

$$f_{1,j}(s) := m_{1,j}(s) m_{2,j}(s), \quad f_{2,j}(s) := m_{2,j}(s) m_{3,j}(s), \quad f_{3,j}(s) := m_{1,j}(s) m_{3,j}(s). \tag{A.16}$$

Thus by Hölder's inequality (A.14) is bounded by

$$\int_{([-\pi,\pi]^3)^3} \mathrm{d}\alpha^{(1)} \mathrm{d}\alpha^{(2)} \mathrm{d}\alpha^{(3)} \prod_{j,\ell=1}^{3} \left(\int_{\mathbb{R}^4} \mathrm{d}s f_{\ell,j}(s)^{-\frac{9}{4}} \right)^{\frac{1}{9}} = \left[\int_{[-\pi,\pi]^3} \mathrm{d}\alpha^{(1)} \left(\prod_{\ell=1}^{3} \int_{\mathbb{R}^4} \mathrm{d}s f_{\ell,1}(s)^{-\frac{9}{4}} \right)^{\frac{1}{9}} \right]^3 . \tag{A.17}$$

The main point of introducing the functions f above is that each of them is a product of two terms, one of which depends only on either R_1 or R_2. Since R_1 does not depend on

s_3 or s_4, and R_2 does not depend on s_1 or s_2, it is possible to estimate the four-dimensional integral over s by a product of two two-dimensional integrals. Even though the above integrals might be infinite for some "bad" choice of $\alpha := \alpha^{(1)}$, we can nevertheless always estimate

$$\prod_{\ell=1}^{3} \int_{\mathbb{R}^4} \mathrm{d}s f_{\ell,1}(s)^{-\frac{9}{4}} \le (G_1(\alpha)G_2(\alpha))^2 G_3(\alpha)G_4(\alpha)\,, \tag{A.18}$$

where

$$G_1(\alpha) := \int \mathrm{d}s_1\mathrm{d}s_2\,(1 + R_1(s,\alpha))^{-\frac{9}{4}}\,, \tag{A.19}$$

$$G_2(\alpha) := \int \mathrm{d}s_3\mathrm{d}s_4\,(1 + R_2(s,\alpha))^{-\frac{9}{4}}\,, \tag{A.20}$$

$$G_3(\alpha) := \sup_{s_3,s_4} \int \mathrm{d}s_1\mathrm{d}s_2\,(1 + R_3(s,\alpha))^{-\frac{9}{4}}\,, \tag{A.21}$$

$$G_4(\alpha) := \sup_{s_1,s_2} \int \mathrm{d}s_3\mathrm{d}s_4\,(1 + R_3(s,\alpha))^{-\frac{9}{4}}\,. \tag{A.22}$$

We will next derive upper bounds for G_i. Only the most complicated case, G_4, will be considered here in detail; the other cases can be estimated similarly. We first recall the definition of R_3 and the lower bound $|z| \ge \max(|\mathrm{Re}\,z|, |\mathrm{Im}\,z|)$ valid for any complex z. Since $|\sigma_2 + \sigma_3 \mathrm{e}^{\mathrm{i}\alpha_2}| \ge |\sin\alpha_2|$, it is nonzero apart from a set of measure zero. Whenever $\sin\alpha_2 \ne 0$, we can estimate

$$\begin{aligned} &R_3(s,\alpha) \\ &\quad = \left| s_1(\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_1}) + s_2(\sigma_2\mathrm{e}^{-\mathrm{i}\alpha_3} + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_1}) + s_3(\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}(\alpha_2-\alpha_3)}) + s_4\mathrm{e}^{-\mathrm{i}\alpha_3}(\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}) \right| \\ &\quad \ge |\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}| \max\left(|s_4 + a(\alpha,s_1,s_2,s_3)|\,, \left| s_3 \mathrm{Im}\,\frac{\sigma_2\mathrm{e}^{\mathrm{i}\alpha_3} + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}}{\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}} + b(\alpha,s_1,s_2) \right| \right)\,, \end{aligned} \tag{A.23}$$

where $a, b \in \mathbb{R}$ depend only on the shown variables. Here

$$\mathrm{Im}\,\frac{\sigma_2\mathrm{e}^{\mathrm{i}\alpha_3} + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}}{\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}} = \mathrm{Im}\,\frac{\sigma_2\mathrm{e}^{\mathrm{i}\alpha_3} - \sigma_2}{\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}} = \sigma_2\sigma_3 \sin\frac{\alpha_3}{2} \mathrm{Im}\,\frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}(\alpha_3-\alpha_2)/2}}{\mathrm{e}^{\mathrm{i}\frac{\alpha_2}{2}} + \sigma_2\sigma_3\mathrm{e}^{-\mathrm{i}\frac{\alpha_2}{2}}}\,. \tag{A.24}$$

If $\sigma_2\sigma_3 = 1$, then $|\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}| = 2|\cos(\alpha_2/2)|$, and we find

$$|\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}| \left| \mathrm{Im}\,\frac{\sigma_2\mathrm{e}^{\mathrm{i}\alpha_3} + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}}{\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}} \right| = 2\left|\sin\frac{\alpha_3}{2}\right| \left|\cos\frac{\alpha_3 - \alpha_2}{2}\right|\,. \tag{A.25}$$

Else, we have $\sigma_2\sigma_3 = -1$, and thus $|\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}| = 2|\sin(\alpha_2/2)|$ and

$$|\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}| \left| \mathrm{Im}\,\frac{\sigma_2\mathrm{e}^{\mathrm{i}\alpha_3} + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}}{\sigma_2 + \sigma_3\mathrm{e}^{\mathrm{i}\alpha_2}} \right| = 2\left|\sin\frac{\alpha_3}{2}\right| \left|\sin\frac{\alpha_3 - \alpha_2}{2}\right|\,. \tag{A.26}$$

Therefore, apart from a set of measure zero

$$R_3(s,\alpha) \geq \max\left(|s'_4 + a'(\alpha, s_1, s_2, s_3)|\,, |s'_3 + b'(\alpha, s_1, s_2)|\right)\,, \tag{A.27}$$

where $a', b' \in \mathbb{R}$, $s'_3 = 2\,|\sin(\alpha_3/2)|\,|t((\alpha_3-\alpha_2)/2)|s_3$ and $s'_4 = 2|t(\alpha_2/2)|s_4$, with $t(x) = \cos(x)$, if $\sigma_2\sigma_3 = 1$, and $t(x) = \sin(x)$, if $\sigma_2\sigma_3 = -1$.

To estimate G_4, we represent the integrand as the product of its square roots, apply the first of the lower bounds implied by (A.27) to the first factor, and the second bound to the second factor. Then we integrate over s_4 first, change it to $s'_4 + a'$, and then change s_3 to $s'_3 + b'$. Since $\int_{\mathbb{R}} \mathrm{d}r(1+|r|)^{-9/8} < \infty$, this shows that there is a pure constant C such that almost everywhere

$$G_4(\alpha) \leq \frac{C}{|t_4(\alpha_2/2)\sin(\alpha_3/2)t_4((\alpha_3-\alpha_2)/2)|}\,, \tag{A.28}$$

where t_4 denotes either "sin" or "cos" depending on the sign of $\sigma_2\sigma_3$.

Repeating analogous steps to the other three cases, yields also the following almost everywhere valid uppers bounds for some pure constant C

$$G_1(\alpha) \leq \frac{C}{|t_1(\alpha_1/2)\sin(\alpha_3/2)t_1((\alpha_1+\alpha_3)/2)|}\,, \tag{A.29}$$

$$G_2(\alpha) \leq \frac{C}{|t_2(\alpha_2/2)\sin(\alpha_3/2)t_2((\alpha_3-\alpha_2)/2)|}\,, \tag{A.30}$$

$$G_3(\alpha) \leq \frac{C}{|t_3(\alpha_1/2)\sin(\alpha_3/2)t_3((\alpha_1+\alpha_3)/2)|}\,, \tag{A.31}$$

where each t_i, $i = 1, 2, 3$, denotes either "sin" or "cos" depending on the value of σ.

Now we can collect the bounds together and apply once more Hölder's inequality to simplify the estimates. We find that

$$\int_{\mathbb{R}^4} \mathrm{d}s|\mathcal{G}(s)| \leq \left(\frac{C_1}{2\pi}\right)^9 \int \mathrm{d}\alpha\,(G_1G_2)^{\frac{1}{3}} \int \mathrm{d}\alpha\,(G_1G_4)^{\frac{1}{3}} \int \mathrm{d}\alpha\,(G_2G_3)^{\frac{1}{3}}\,. \tag{A.32}$$

which is finite, since each of the three integrals is finite. For instance, by (A.29) and (A.30),

$$\begin{aligned}
&\int \mathrm{d}\alpha\,(G_1G_2)^{\frac{1}{3}} \\
&\quad\leq C^{\frac{2}{3}} \int \mathrm{d}\alpha_3\,|\sin(\alpha_3/2)|^{-\frac{2}{3}} \\
&\qquad\times \int \mathrm{d}\alpha_1\,|t_1(\alpha_1/2)t_1((\alpha_1+\alpha_3)/2)|^{-\frac{1}{3}} \int \mathrm{d}\alpha_2\,|t_2(\alpha_2/2)t_2((\alpha_3-\alpha_2)/2)|^{-\frac{1}{3}} \\
&\quad\leq C^{\frac{2}{3}} \int \mathrm{d}\alpha_3\,|\sin(\alpha_3/2)|^{-\frac{2}{3}} \int \mathrm{d}\alpha_1\,|t_1(\alpha_1/2)|^{-\frac{2}{3}} \int \mathrm{d}\alpha_2\,|t_2(\alpha_2/2)|^{-\frac{2}{3}}\,,
\end{aligned} \tag{A.33}$$

and all of the remaining integrals contain only integrable singularities of the form $r^{-2/3}$, for all allowed choices of t_1 and t_2. The integrals $\int \mathrm{d}\alpha\,(G_1G_4)^{\frac{1}{3}}$ and $\int \mathrm{d}\alpha\,(G_2G_3)^{\frac{1}{3}}$ are of the same form, and the argument proving their finiteness is identical to the above. This completes the proof of the Proposition. □

References

[1] L. W. Nordheim, *On the kinetic method in the new statistics and its application in the electron theory of conductivity*, Proc. R. Soc. Lond. Ser. A **119** (1928) 689–698.

[2] R. Peierls, *Zur kinetischen Theorie der Wärmeleitung in Kristallen*, Ann. Phys. **3** (1929) 1055–1101.

[3] C. Villani, *A review of mathematical topics in collisional kinetic theory*. In S. J. Friedlander and D. Serre (eds.), *Handbook of Mathematical Fluid Dynamics*, volume 1. Elsevier, Amsterdam, 2002.

[4] J. Dolbeault, *Kinetic models and quantum effects: A modified Boltzmann equation for Fermi-Dirac particles*, Arch. Ration. Mech. Anal. **127** (1994) 101–131.

[5] P.-L. Lions, *Compactness in Boltzmann's equation via Fourier integral operators and applications, III*, J. Math. Kyoto Univ. **34** (1994) 539–584.

[6] X. Lu, *On isotropic distributional solutions to the Boltzmann equation for Bose-Einstein particles*, J. Stat. Phys. **116** (2004) 1597–1649.

[7] X. Lu, *The Boltzmann equation for Bose-Einstein particles: Velocity concentration and convergence to equilibrium*, J. Stat. Phys. **119** (2005) 1027–1067.

[8] M. Escobedo, S. Mischler, and J. J. L. Velázquez, *Singular solutions for the Uehling-Uhlenbeck equation*, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008) 67–107.

[9] M. Escobedo and J. J. L. Velázquez, *Finite time blow-up for the bosonic Nordheim equation*, preprint (2012), `http://arxiv.org/abs/1206.5410`

[10] M. Escobedo and J. J. L. Velázquez, *On the blow up of supercritical solution of the Nordheim equation for bosons*, preprint (2012), `http://arxiv.org/abs/1210.1664`

[11] M. L. R. Fürst, J. Lukkarinen, P. Mei, and H. Spohn, *Formal derivation of the matrix-valued Boltzmann equation for the Hubbard model*. In preparation.

[12] J. Lukkarinen and H. Spohn, *Weakly nonlinear Schrödinger equation with random initial data*, Invent. Math. **183** (2011) 79–188.

[13] W. Rudin, *Functional Analysis*. Tata McGraw-Hill, New Delhi, 1974.

[14] H. Araki and S. Yamagami, *An inequality for Hilbert-Schmidt norm*, Commun. Math. Phys. **81** (1981) 89–96.

[15] E. Seiler and B. Simon, *An inequality among determinants*, Proc. Nat. Acad. Sci. USA **72** (1975) 3277–3278.

[16] I. Gohberg, I. Goldberg, and N. Krupnik, *Traces and Determinants of Linear Operators*. Birkhäuser, Berlin, 2000.

[17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*. Academic Press, New York, 1975.

[18] R. Bellman, *Introduction to Matrix Analysis*. SIAM, Philadelphia, 2nd edition, 1970.

[19] W. Rudin, *Real and Complex Analysis*. McGraw-Hill, New York, 3rd edition, 1987.